1. Differentiable Functions

A real-valued function of a vector variable, $F(\mathbf{x})$ is *differentiable at a point* $\mathbf{a}$ if and only if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet \mathbf{h}}{|\mathbf{h}|} = 0$$

As we say in the section on limits, if the limit exists, then its value can be calculated. To do this, fix a vector $\mathbf{h} \neq \mathbf{0}$ and let $t\mathbf{h}$ be a line to $\mathbf{0}$. Since there is an $|\mathbf{h}|$ in our formula it is useful to compute $\lim\limits_{t\to 0^{\pm}}$.

If f is differentiable, the limit is 0 along every line so

$$0 = \lim_{t\to 0^+} \frac{f(\mathbf{a}+t\mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet (t\mathbf{h})}{t\,|\mathbf{h}|}$$

or

$$\lim_{t\to 0^+} \frac{f(\mathbf{a}+t\mathbf{h}) - f(\mathbf{a})}{t\,|\mathbf{h}|} = \nabla f(\mathbf{a}) \bullet \frac{\mathbf{h}}{|\mathbf{h}|}$$

Similalrly

$$\lim_{t\to 0^-} \frac{f(\mathbf{a}+t\mathbf{h}) - f(\mathbf{a})}{-t\,|\mathbf{h}|} = -\nabla f(\mathbf{a}) \bullet \frac{\mathbf{h}}{|\mathbf{h}|}$$

so

$$(*) \qquad \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{h}) - f(\mathbf{a})}{t\,|\mathbf{h}|} = \nabla f(\mathbf{a}) \bullet \frac{\mathbf{h}}{|\mathbf{h}|}$$

If we let $\mathbf{h} = \mathbf{e}_i$, $(*)$ shows that

$$\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a})}{t} = \frac{\partial f}{\partial x_i}(\mathbf{a})$$

and therefore, if f is differentiable at $\mathbf{a}$, all the partials exist at $\mathbf{a}$.

The existence of partial derivatives is insufficient to make the function differentiable, just as the existence of limits along some lines in insufficient to guarantee the limit exists.

Theorem. *If a function has all partial derivatives continuous in a neighborhood of* $\mathbf{a}$ *then* f *is differentiable in that neighborhood.*

As a practical matter, write down the partials and argue that they are continuous using your theorems on when a function is continuous.

1.1. Directional Derivatives.

By definition, if $\mathbf{u}$ is a unit vector, the *directional derivative of* f *in the direction* $\mathbf{u}$ *at the point* $\mathbf{a}$ is

$$D_{\mathbf{u}}(f)(\mathbf{a}) = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t}$$

It follows that if f is differentiable at $\mathbf{a}$,

$$D_{\mathbf{u}}(f)(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \mathbf{u}$$

1.2. **Properties of the Gradient.**

- $D_{\mathbf{u}}(f)(\mathbf{a})$ is the instantaneous rate of change of f at $\mathbf{a}$ in the direction $\mathbf{u}$. If $D_{\mathbf{u}}(f)(\mathbf{a})$ is positive, the value of f is increasing; if $D_{\mathbf{u}}(f)(\mathbf{a})$ is negative, the value of f is decreasing.
- The direction of fastest increase in f at a point $\mathbf{a}$ is the direction of the gradient. (Or meaningless if $\nabla f(\mathbf{a}) = \mathbf{0}$.) The value of this rate is $|\nabla f(\mathbf{a})|$.
- The direction of fastest decrease in f at a point $\mathbf{a}$ is the negative of the direction of the gradient. (Or meaningless if $\nabla f(\mathbf{a}) = \mathbf{0}$.) The value of this rate is $-|\nabla f(\mathbf{a})|$.
- In two variables, if $\mathbf{r}(t)$ is a smooth parametrization of a level curve of $f(x, y)$ then $\mathbf{r}'(t_0)$ is perpendicular to the gradient $\nabla f\big(\mathbf{r}(t_0)\big)$.
- In three variables, if $\mathbf{r}(t)$ is a smooth parametrization of a curve lying in the surface $f(x, y, z) = c$ then $\mathbf{r}'(t_0)$ is perpendicular to the gradient $\nabla f\big(\mathbf{r}(t_0)\big)$.

2. More on level curves

Suppose $f(x, y) = c$ is a level curve. Then at any point $\mathbf{a}$ on the level curve, $\nabla f(\mathbf{a})$ is orthogonal to the level curve. Hence as long as $\nabla f(\mathbf{a}) \neq \mathbf{0}$, $\mathbf{N} = \dfrac{\nabla f(\mathbf{a})}{|\nabla f(\mathbf{a})|}$ is a unit normal to the curve and this normal is the direction of fastest increase in the value of f.

It is also possible to work out the remaining invariants of a plane curve for this sort of example. In particular, $\mathbf{B} = \langle 0, 0, 1\rangle$ and so a vector in the direction of $\mathbf{T}$ is $\mathbf{N}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left\langle \dfrac{\partial f}{\partial y}, -\dfrac{\partial f}{\partial x}, 0 \right\rangle$.

The remaining invariant is the curvature. Suppose $\left.\dfrac{\partial f}{\partial x}\right|_{\mathbf{x}=\mathbf{a}=\langle x_0, y_0\rangle} \neq 0$. Then the Implicit Function Theorem says that near this point on the level curve, there exists a differentiable function $\alpha(x)$ such that $\mathbf{r} = \langle x, \alpha(x)\rangle$ is the level curve. The curvature is $\dfrac{|\alpha''(x_0)|}{\left(\sqrt{1 + \left(\alpha'(x_0)\right)^2}\right)^3}$.

By the Chain Rule, $\nabla f(\langle x, \alpha(x)\rangle) \bullet \langle 1, \alpha'(x)\rangle = 0$ so

$$\frac{\partial f}{\partial x}\left(\langle x, \alpha(x)\rangle\right) + \frac{\partial f}{\partial y}\left(\langle x, \alpha(x)\rangle\right)\alpha'(x) = 0$$

$$\frac{\partial^2 f}{\partial x\,\partial x} + \frac{\partial^2 f}{\partial y\,\partial x}\alpha'(x) + \left(\frac{\partial^2 f}{\partial y\,\partial x} + \frac{\partial^2 f}{\partial y\,\partial y}\alpha'(x)\right)\alpha'(x) + \frac{\partial f}{\partial y}\left(\langle x, \alpha(x)\rangle\right)\alpha''(x) = 0 \tag{1}$$

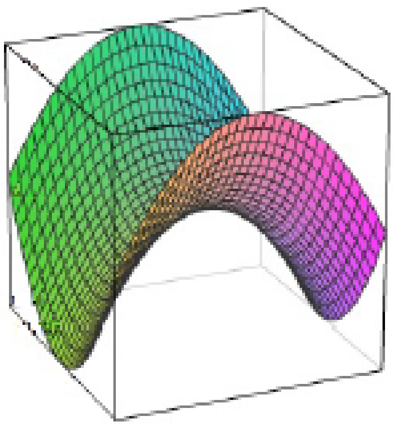
Multiply (1) by $\left(\dfrac{\partial f}{\partial y}\right)^2$ and assume mixed partials are equal:

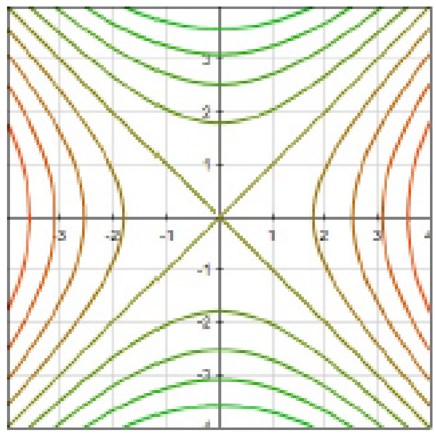
$$\frac{\partial^2 f}{\partial x\,\partial x}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial y\,\partial x}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y\,\partial y}\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^3\alpha''(x) = 0 \tag{2}$$

and so

$$\kappa(x_0, y_0) = \frac{\left| \frac{\partial^2 f}{\partial x \partial x} \left(\frac{\partial f}{\partial y} \right)^2 - 2 \frac{\partial^2 f}{\partial y \partial x} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y \partial y} \left(\frac{\partial f}{\partial x} \right)^2 \right|}{\left(\sqrt{\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2} \right)^3}$$

Consider this example of the level curves of $f(x, y) = x^2 - y^2$ from worksheet 4.

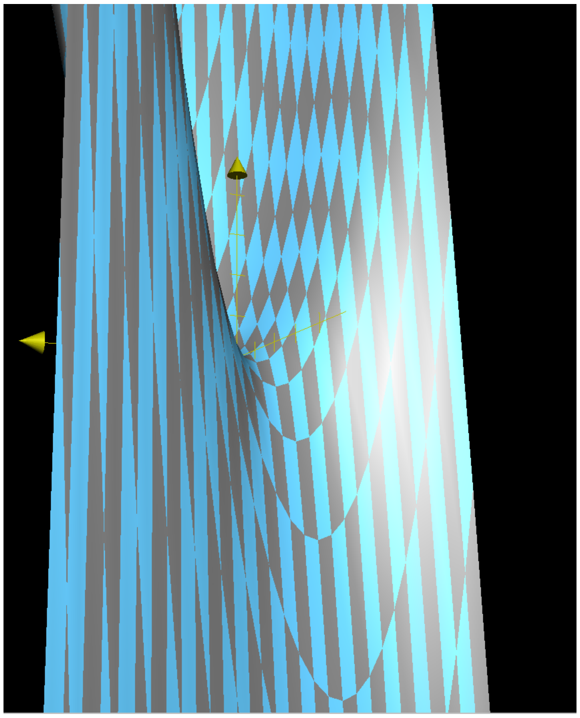


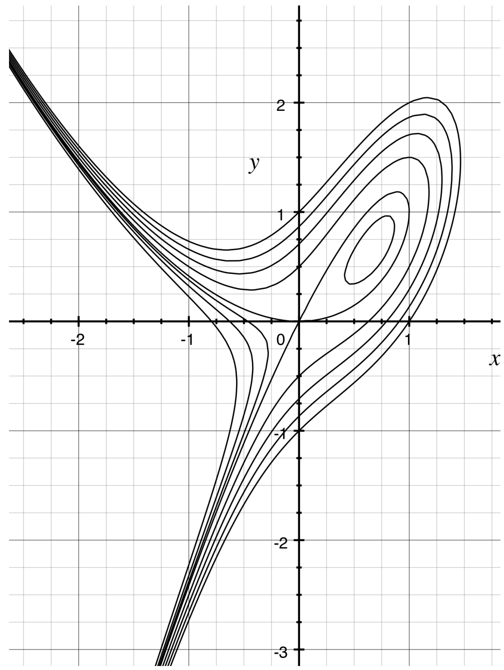


Here $\nabla f(x) = \langle 2x, -2y \rangle$ and $\begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ so

$$\kappa(x, y) = \frac{|x^2 - y^2|}{\left(x^2 + y^2 \right)^{\frac{3}{2}}}$$

A second example: the graph and level curves of $f(x,y) = x^3 - 2xy + y^2$





Here $\nabla f(x) = \langle 3x^2 - 2y, -2x + 2y \rangle$ and $\begin{bmatrix} \dfrac{\partial^2 f}{\partial x \partial x} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} 6x & -2 \\ -2 & 2 \end{bmatrix}$ so

$$\kappa(x,y) = \frac{|6x(-2x+2y)^2 - 8(3x^2-2y)(-2x+2y) + 2(3x^2-2y)^2|}{\big((3x^2-2y)^2 + (-2x+2y)^2\big)^{\frac{3}{2}}}$$

3. The curve of intersection of two implicit surfaces

An implicit surface is the set of solutions to an equation $f(x,y,z) = c$.

Given two implicit surfaces $f(x,y,z) = c$ and $g(x,y,z) = d$ they typically intersect in a curve. If $\mathbf{a}$ is a point on both surfaces we can try to compute the typical invariants of the intersect curve at this point.

If we parametrize the curve by $\mathbf{r}(t)$ then $f\big(\mathbf{r}(t)\big) = c$ and $\nabla f\big(\mathbf{r}(t)\big) \bullet \mathbf{r}'(t) = 0$. In particular $\mathbf{r}'(t)$ is perpendicular to $\nabla f\big(\mathbf{r}(t)\big)$. Similarly $\mathbf{r}'(t)$ is perpendicular to $\nabla g\big(\mathbf{r}(t)\big)$. Hence we will take $\mathbf{r}'(t)$ parallel to $\nabla f\big(\mathbf{r}(t)\big) \times \nabla g\big(\mathbf{r}(t)\big)$ and so

$$\mathbf{T}(\mathbf{a}) = \frac{\nabla f\big(\mathbf{a}\big) \times \nabla g\big(\mathbf{a}\big)}{\big|\nabla f\big(\mathbf{a}\big) \times \nabla g\big(\mathbf{a}\big)\big|}$$

If we assume $\nabla f\big(\mathbf{a}\big) \times \nabla g\big(\mathbf{a}\big) \neq \mathbf{0}$ it follows that $\nabla f\big(\mathbf{a}\big) \neq \mathbf{0}$ and $\nabla g\big(\mathbf{a}\big) \neq \mathbf{0}$.

The Implicit Function Theorem says that around $\mathbf{a}$, $f(x,y,z) = c$ is a graph $z = \alpha(x,y)$ and $g(x,y,z) = d$ is a graph $z = \beta(x,y)$.

The intersection of the two surfaces is the set of all points where $\alpha(x, y) = \beta(x, y)$ and (under some non-zero hypothesis of partials) there is a function $\gamma(x)$ so that $\alpha\big(x, \gamma(x)\big) = \beta\big(x, \gamma(x)\big)$. The curve is paramterized by

$$\mathbf{r}(x) = \big\langle x, \gamma(x), \alpha\big(x, \gamma(x)\big)\big\rangle = \big\langle x, \gamma(x), \beta\big(x, \gamma(x)\big)\big\rangle$$

$f\big(\langle x, y, \alpha(x, y)\rangle\big) = c$

$\nabla f\big(\langle x, y, \alpha(x, y)\rangle\big) \bullet \left\langle 1, 0, \dfrac{\partial\,\alpha}{\partial\,x}\right\rangle = 0$

$\nabla f\big(\langle x, y, \alpha(x, y)\rangle\big) \bullet \left\langle 0, 1, \dfrac{\partial\,\alpha}{\partial\,y}\right\rangle = 0$ so

$$\left\langle 1, 0, \frac{\partial\,\alpha}{\partial\,x}\right\rangle \times \left\langle 0, 1, \frac{\partial\,\alpha}{\partial\,y}\right\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial\,\alpha}{\partial\,x} \\ 0 & 1 & \dfrac{\partial\,\alpha}{\partial\,y} \end{vmatrix} = \begin{vmatrix} 0 & \dfrac{\partial\,\alpha}{\partial\,x} \\ 1 & \dfrac{\partial\,\alpha}{\partial\,y} \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & \dfrac{\partial\,\alpha}{\partial\,x} \\ 0 & \dfrac{\partial\,\alpha}{\partial\,y} \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$

$$= \left\langle -\frac{\partial\,\alpha}{\partial\,x}, -\frac{\partial\,\alpha}{\partial\,y}, 1\right\rangle$$

is parallel to ∇f so

$$\nabla f = \left\langle -\frac{\partial\,\alpha}{\partial\,x} \cdot \frac{\partial\,f}{\partial\,z}, -\frac{\partial\,\alpha}{\partial\,y} \cdot \frac{\partial\,f}{\partial\,z}, \frac{\partial\,f}{\partial\,z}\right\rangle \quad \text{so} \quad \frac{\partial\,\alpha}{\partial\,x} = -\frac{\dfrac{\partial\,f}{\partial\,x}}{\dfrac{\partial\,f}{\partial\,z}} \quad \text{and} \quad \frac{\partial\,\alpha}{\partial\,y} = -\frac{\dfrac{\partial\,f}{\partial\,y}}{\dfrac{\partial\,f}{\partial\,z}}$$

$$\nabla g = \left\langle -\frac{\partial\,\beta}{\partial\,x} \cdot \frac{\partial\,g}{\partial\,z}, -\frac{\partial\,\beta}{\partial\,y} \cdot \frac{\partial\,g}{\partial\,z}, \frac{\partial\,g}{\partial\,z}\right\rangle \quad \text{so} \quad \frac{\partial\,\beta}{\partial\,x} = -\frac{\dfrac{\partial\,g}{\partial\,x}}{\dfrac{\partial\,g}{\partial\,z}} \quad \text{and} \quad \frac{\partial\,\beta}{\partial\,y} = -\frac{\dfrac{\partial\,g}{\partial\,y}}{\dfrac{\partial\,g}{\partial\,z}}$$

Since $\alpha\big(x, \gamma(x)\big) = \beta\big(x, \gamma(x)\big)$, differentiating with respect to x gives $\nabla\alpha \bullet \langle 1, \gamma'\rangle = \nabla\beta \bullet \langle 1, \gamma'\rangle$ so

$$\gamma'(x) = \frac{\dfrac{\partial\,\alpha}{\partial\,x} - \dfrac{\partial\,\beta}{\partial\,x}}{\dfrac{\partial\,\beta}{\partial\,y} - \dfrac{\partial\,\alpha}{\partial\,y}} = \frac{\dfrac{\partial\,g}{\partial\,x} - \dfrac{\partial\,f}{\partial\,x}}{\dfrac{\partial\,f}{\partial\,y} - \dfrac{\partial\,g}{\partial\,y}}$$

With these formulas and the Chain Rule one can calculate $\mathbf{r}'$, $\mathbf{r}''$ and $\mathbf{r}'''$ in terms of the (higher order) partials of f and g and hence obtain the curvature, the torsion and the Frenet-Serret frame at a point on the intersection curve.

4. Approximations (Extra)

Recall that a function is *differentiable* if and only if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\left|\big(f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})\big)-\nabla f(\mathbf{a})\bullet\mathbf{h}\right|}{|\mathbf{h}|}=0$$

This in turn means that if $|\mathbf{h}|$ is small,

$$(*) \qquad \big(f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})\big)-\nabla f(\mathbf{a})\bullet\mathbf{h}\approx 0$$

4.1. **Approximation values.** Suppose you have a function $f(\mathbf{x})$ and you know $f(\mathbf{a})$ but you want to know $f(\mathbf{b})$ for some other input $\mathbf{b}$. Write

$$f(\mathbf{b})=f\big(\mathbf{a}+(\mathbf{b}-\mathbf{a})\big)$$

Then, letting $\mathbf{h}=\mathbf{b}-\mathbf{a}$ and rewriting $(*)$

$$f(\mathbf{b})=f\big(\mathbf{a}+(\mathbf{b}-\mathbf{a})\big)\approx f(\mathbf{a})+\nabla f(\mathbf{a})\bullet(\mathbf{b}-\mathbf{a})$$

4.2. **Solve equations.** Try to solve $f(\mathbf{x})=0$. Pick a point $\mathbf{a}_0$ and evaluate $f(\mathbf{a}_0)$. If this is not 0 solve

$$f(\mathbf{a}_0)+t_0\nabla f(\mathbf{a}_0)\bullet\nabla f(\mathbf{a}_0)=0$$

for t_0. As long as $\nabla f(\mathbf{a}_0)\neq\mathbf{0}$ you can do this.
Since $f(\mathbf{a}_0)+t_0\nabla f(\mathbf{a}_0)\bullet\nabla f(\mathbf{a}_0)=0$, if we take

$$\mathbf{a}_1=\mathbf{a}_0+t_0\nabla f(\mathbf{a}_0)$$

and approximate $f(\mathbf{a}_1)$ we get

$$f(\mathbf{a}_1)\approx f(\mathbf{a}_0)+t_0\nabla f(\mathbf{a}_0)\bullet\nabla f(\mathbf{a}_0)=0$$

So we might expect that $\mathbf{a}_1$ is a better approximation to the solution than was our initial guess $\mathbf{a}_1$. Now iterate.
In summary:

Pick $\mathbf{a}_0$, compute

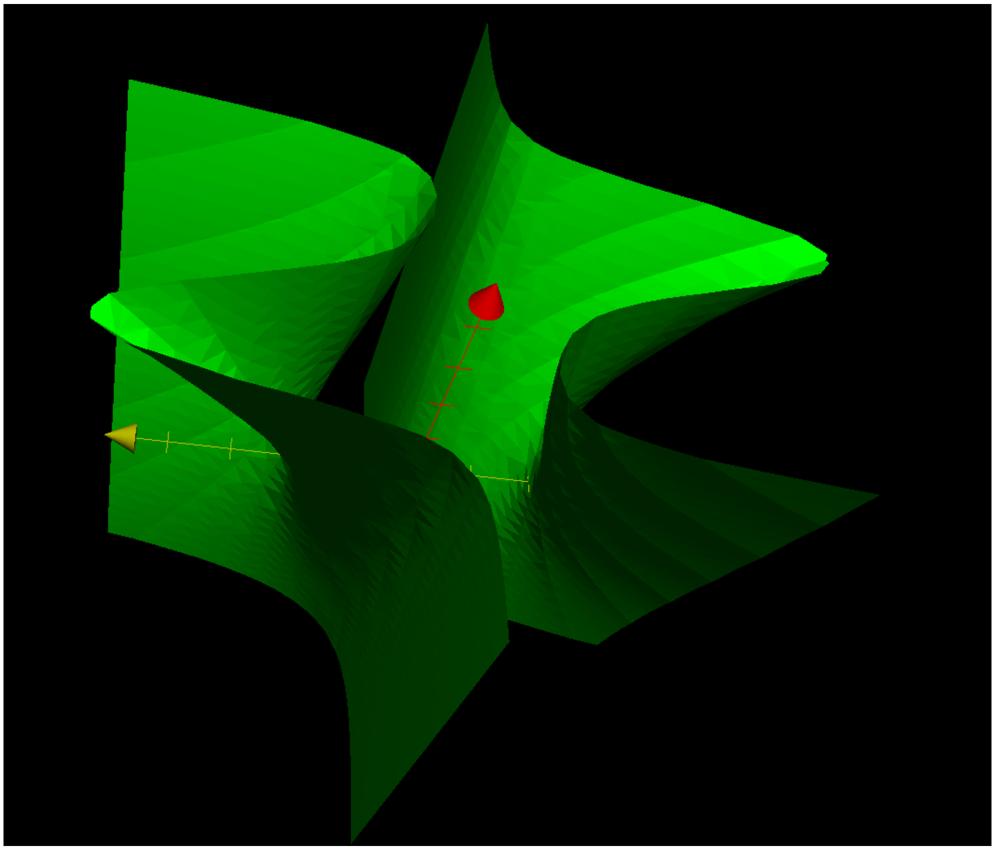
$$\mathbf{a}_1=\mathbf{a}_0-\frac{f(\mathbf{a}_0)}{\nabla f(\mathbf{a}_0)\bullet\nabla f(\mathbf{a}_0)}\nabla f(\mathbf{a}_0)$$

and iterate.
Example: Find a solution to $f(x,y,z)=0$ where

$$f(x,y,z)=x^2+xyz-xz^3-4$$

Compute $\nabla f=\langle 2x+yz-z^3, xz, xy-3xz^2\rangle$.

	x	y	z	f(x,y,z)	f_x(x,y,z)	f_y(x,y,z)	f_z(x,y,z)	f/Grad•Grad
a_0	1	2	3	-24	-19	3	-25	-0.024120603
a_1	0.5417085427	2.072361809	2.3969849246	-8.4760490463	-7.7211278229	1.2984672104	-8.21460289	-0.0658172711
a_2	0.0335249794	2.1578233775	1.8563221791	-4.0790391515	-2.3240947064	0.0622331629	-0.2742334171	-0.7442832331
a_3	-1.6962597426	2.2041424771	1.6522148448	0.3505362172	-4.2610416023	-2.8025855273	10.1526220846	0.00271551
a_4	-1.6846888414	2.2117529262	1.6246452978	0.008853864	-4.0642700231	-2.7370218045	9.6139533387	7.60395100E-005
a_5	-1.6843797963	2.211961048	1.6239142575	6.13224384871103E-006	-4.0591447638	-2.7352883663	9.5998388293	5.28116489E-008
a_6	-1.684379582	2.2119611925	1.6239137505	2.94875235340442E-012	-4.059141211	-2.7352871642	9.5998290437	2.53950719E-014
a_7	-1.684379582	2.2119611925	1.6239137505	0	-4.059141211	-2.7352871642	9.5998290437	0



4.3. **Points of intersection of two implicit surfaces.** Suppose you want a point on $f(x,y,z) = c$ and $g(x,y,z) = d$. Solve

$$\big(f(x,y,z) - c\big)^2 + \big(g(x,y,z) - d\big)^2 = 0$$