## 1. Tangent planes and normal lines

Suppose $f(x, y, z)$ is differentiable and $f(x, y, z) = c$ is an implicitly defined surface. (The book also calls it a level surface. It is a level surface of the function $f(x, y, z) - t$.) Suppose also that $(x_0, y_0, z_0)$ is a point on this surface. Then the *tangent plane* to this surface at the point $(x_0, y_0, z_0)$ is given by

$$(TP) \qquad \nabla f(x_0, y_0, z_0) \bullet \langle x, y, z\rangle = \nabla f(x_0, y_0, z_0) \bullet \langle x_0, y_0, z_0\rangle$$

or if you prefer

$$\nabla f(x_0, y_0, z_0) \bullet \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

Then the *normal line* to this surface at the point $(x_0, y_0, z_0)$ is given by

$$(NL) \qquad \nabla f(x_0, y_0, z_0)t + \langle x_0, y_0, z_0\rangle$$

Just as a line has many equations, a plane does also. Any equation of the form $\mathbf{N} \bullet \langle x, y, z\rangle = c$ with $\mathbf{N} \neq \mathbf{0}$ is a plane. The vector $\mathbf{N}$ is perpendicular to the plane.

Conversely, given $\mathbf{N} \neq \mathbf{0}$, $\mathbf{N} \bullet \langle x, y, z\rangle = \mathbf{N} \bullet \langle x_0, y_0, z_0\rangle$ is a plane with $\mathbf{N}$ as a normal vector and $\langle x_0, y_0, z_0\rangle$ as a point on the plane.

## 2. Gradient formulas

Here are some formulas for computing gradients. They are seldom necessary for computing with formulas but often very important for computing with limited information.

- $\nabla(au + bw) = a\nabla u + b\nabla w$
- $\nabla(uw) = (\nabla u)w + u\nabla w$
- $\nabla\left(\dfrac{u}{w}\right) = \dfrac{1}{w^2}\big((\nabla u)w - u\nabla w\big)$
- $\nabla(u^r) = ru^{r-1}(\nabla u)$

**Example.** If $\nabla u(1, 2, 3) = \langle 5, 3, 2\rangle$, $u(1, 2, 3) = 2$, $\nabla w(1, 2, 3) = \langle 1, -2, 6\rangle$, $w(1, 2, 3) = -5$ find $\nabla(uw)(1, 2, 3)$. Then $(uw)(1, 2, 3) = 2(-5) = -10$ and

$$\nabla(uw)(1, 2, 3) = \langle 5, 3, 2\rangle (-5) + 2 \langle 1, -2, 6\rangle = \langle -23, -19, 2\rangle$$

## 3. Higher directional derivatives

Given a unit vector $\mathbf{v}$ and a function $f(\mathbf{x})$, $D_{\mathbf{v}}f(\mathbf{x})$ is also a real-valued function of $\mathbf{x}$. Hence given another unit vector $\mathbf{w}$, we can compute $D_{\mathbf{w}}\big(D_{\mathbf{v}}f(\mathbf{x})\big)$.

Suppose $\mathbf{x} = \langle x, y\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$ and $\mathbf{w} = \langle w_1, w_2\rangle$. Then $D_{\mathbf{v}}f(\langle x, y\rangle) = \nabla f(\langle x, y\rangle) \bullet \mathbf{u} = \dfrac{\partial f}{\partial x}u_1 + \dfrac{\partial f}{\partial y}u_2$ and then

$$
\begin{aligned}
D_{\mathbf{w}}\big(D_{\mathbf{v}}f(\langle x,y\rangle)\big) =& D_{\mathbf{w}}\left(\frac{\partial f}{\partial x}v_1 + \frac{\partial f}{\partial y}v_2\right)\\
=& \left(\frac{\partial^2 f}{\partial x\,\partial x}v_1 + \frac{\partial^2 f}{\partial x\,\partial y}v_2\right)w_1 + \left(\frac{\partial^2 f}{\partial y\,\partial x}v_1 + \frac{\partial^2 f}{\partial y\,\partial y}v_2\right)w_2\\
=& \frac{\partial^2 f}{\partial x\,\partial x}v_1w_1 + \frac{\partial^2 f}{\partial x\,\partial y}(v_1w_2 + v_2w_1) + \frac{\partial^2 f}{\partial y\,\partial y}v_2w_2
\end{aligned}
$$

This last equation assumes that the mixed partials, $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$, are equal. Given this note

$$
D_{\mathbf{w}}\big(D_{\mathbf{v}}f(\mathbf{x})\big) = D_{\mathbf{v}}\big(D_{\mathbf{w}}f(\mathbf{x})\big)
$$

**Exercise.** Check that in 3 variables

$$
\begin{aligned}
D_{\mathbf{w}}\big(D_{\mathbf{v}}f(\langle x,y,z\rangle)\big) =& \frac{\partial^2 f}{\partial x\,\partial x}v_1w_1 + \frac{\partial^2 f}{\partial y\,\partial y}v_2w_2 + \frac{\partial^2 f}{\partial z\,\partial z}v_3w_3 +\\
& \frac{\partial^2 f}{\partial x\,\partial y}(v_1w_2 + v_2w_1) + \frac{\partial^2 f}{\partial x\,\partial z}(v_1w_3 + v_3w_1) + \frac{\partial^2 f}{\partial y\,\partial z}(v_2w_3 + v_3w_2)
\end{aligned}
$$

**Exercise.** Go back to the level curve section last time and recall that for the level curve $f(x,y) = c$ and the point $(x_0,y_0)$, $\nabla f(x_0,y_0)$ is a perpendicular to the tangent line at the point. We wrote down a unit tangent vector $\mathbf{T}(x_0,y_0) = \dfrac{\left\langle \frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right\rangle}{\left|\left\langle \frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right\rangle\right|}$. Show that our formula for the curvature at $(x_0,y_0)$ is

$$
\kappa(x_0,y_0) = \frac{\Big|D_{\mathbf{T}}\big(D_{\mathbf{T}}f(x,y)\big)\big|_{(x_0,y_0)}\Big|}{|\nabla f(x_0,y_0)|}
$$