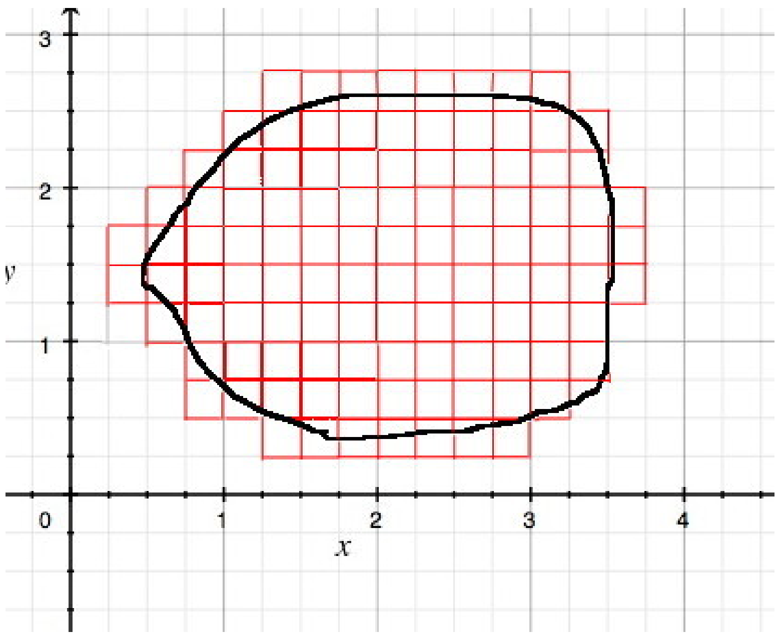
1. Definition

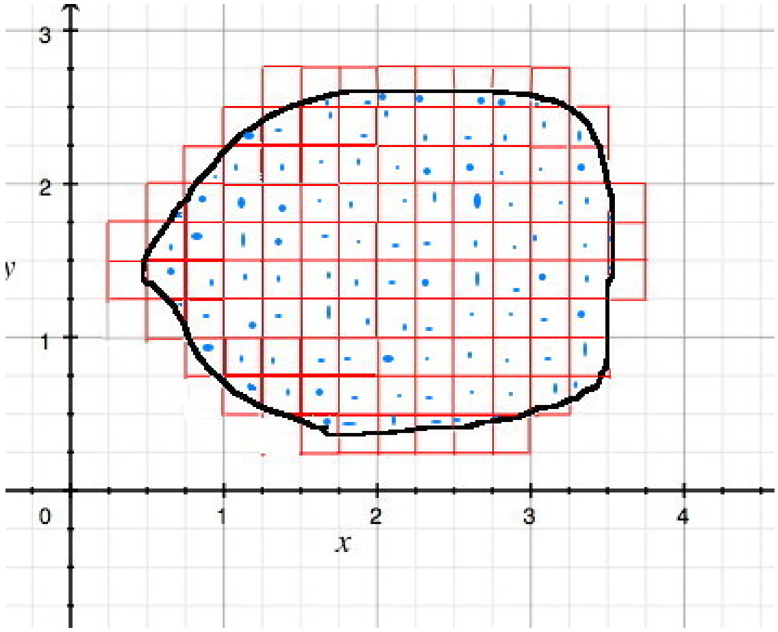
Given a region R in the plane, a *partition of* R consists of a finite set of regions $\mathcal{P} = \{P_1, \cdots, P_n\}$ such that the P_i cover R, or equivalently $R \subset \bigcup_{i=1}^{n} P_i$. Furthermore, for each P_i, R intersects P_i or $P_i \cap R \neq \emptyset$ and $P_i \cap P_j$ for $i \neq j$ is a curve of some sort. We also want the boundary of R to be a finite collection of curves.

A *mesh of* $\mathcal{P}$ is a number m such that each P_i is contained in a ball of radius m (the center of this ball can change for each P_i).



In this picture, each P_i is a red square, 0.25 units on a side. If you center your balls at the centers of the squares you see a mesh is $\sqrt{\dfrac{0.25}{2}}$. It is not hard to convince yourself that this is the smallest possible mesh for this partition.

Next pick a point (x_i, y_i) in each $P_i \cap R$.



In this picture, each p_i is sort of a blue dot. Then the Riemann sum associated to the partition $\mathcal{P}$, the points p_i and the function $f(x, y)$ is

$$\mathrm{RS}(\mathcal{P}, \{p_i\}, f) = \sum_{i=1}^{n} f(x_i, y_i) \cdot \mathrm{area}(P_i)$$

$$\iint_R f(x, y)dA = \lim_{\mathrm{mesh}\to 0} \mathrm{RS}(\mathcal{P}, \{p_i\}, f)$$

provided this limit exists.

A basic result due to Riemann is the following

If f is continuous on a nice region R and if R is closed and bounded, then $\iint_R f(x, y)\, dA$ exists.

Other basic results which follow from the definition.

If f and g satisfy $f \leqslant g$ on R and if $\iint_R f(x, y)dA$ and $\iint_R g(x, y)dA$ exist, then

$$\iint_R f(x, y)dA \leqslant \iint_R g(x, y)dA$$

The integrals are equal if and only if the functions are equal.

A corollary of this result is that if $m \leqslant f(x, y) \leqslant M$ on R then

$$m \cdot \mathrm{area}(R) \leqslant \iint_R f(x, y)\, dA \leqslant M \cdot \mathrm{area}(R)$$

provided $\iint_R f(x, y)\, dA$ exists.

A special case of this inequality is

$$\iint_R 1\, dA = \text{area}(R)$$

Additional applications which we will explore later are:

- If μ is the density function of a thin plate R, $\displaystyle\iint_R \mu(x,y)\, dA = \text{mass}(R)$.
- If μ is the density function of a thin plate R, $\displaystyle\iint_R x\mu(x,y)\, dA = \text{moment about the } y\text{-axis}(R)$.
- If μ is the density function of a thin plate R, $\displaystyle\iint_R y\mu(x,y)\, dA = \text{moment about the } x\text{-axis}(R)$.
- Moments of inertia, volumes, ...

If f and g are defined on R and if $\displaystyle\iint_R f(x,y)dA$ and $\displaystyle\iint_R g(x,y)dA$ exist, then

$$\iint_R \big(f(x,y)+g(x,y)\big)\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

If f is defined on R, if c is a constant, and if $\displaystyle\iint_R f(x,y)\, dA$ exists, then

$$\iint_R c\cdot f(x,y)\, dA = c\iint_R f(x,y)\, dA$$

If $R = R_1 \cup R_2$ and if $R_1 \cap R_2$ is contained in a curve, then

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

provided $\displaystyle\iint_{R_1} f(x,y)\, dA$ and $\displaystyle\iint_{R_2} f(x,y)\, dA$ exist.

Midpoint Rule: If R is partitioned into rectangles $x_0 < x_1 < \cdots < x_n$ and $y_0 < y_1 < \cdots < y_m$ then

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{x}_i,\bar{y}_j)\cdot |x_i - x_{i-1}|\cdot|y_j - y_{j-1}|$$

where $\bar{x}_i = \dfrac{x_i + x_{i-1}}{2}$ and $\bar{y}_j = \dfrac{y_j + y_{j-1}}{2}$.

Average Value By definition, the average value of a function f on a region R is

$$\frac{1}{\text{area}(R)}\iint_R f(x,y)\, dA$$

Volume of a Solid: If the graph of $z = f(x,y)$ lies above the xy-plane for (x,y) is some closed bounded region R, then the volume of the solid lying above the xy-plane, below the graph of $z = f(x,y)$ and inside the cylinder over R is given by

$$\iint_R f(x,y)\, dA$$

This is the analogue of the first year calculus result that an area is given by a definite integral.

2. Iterated integrals

An iterated (double) integral is an expression of the form

$$\int_a^b \int_{b(x)}^{t(x)} f(x,y)\,dy\,dx$$

There will be other variations as we go along. One obvious one is to switch the roles of x and y but we can replace both variables by any other pair of names of variables.

The new idea in this chapter is that we can evaluate a *double integral* $\iint_A f(x,y)\,dA$ by doing an appropriate *iterated integral* $\int_a^b \int_{b(x)}^{t(x)} f(x,y)\,dy\,dx$. Because of the Riemann sum definition of the double integral, many questions have a double integral as an answer.

The *outer integral* (the $\int_a^b \cdots dx$ is a first year calculus definite integral as we shall see, so the new material is to work out what

$$\int_{b(x)}^{t(x)} f(x,y)\,dy$$

is. Here $b(x)$ and $t(x)$ are functions of x where usually $b(x) \leqslant t(x)$ for $x \in [a,b]$ but the inequality is not necessary to work out the answer so don't bother checking it at this point. All you do is do a first year calculus definite integral treating x as a constant.

In more detail, by the Fundamental Theorem of Calculus, you need to find $F(x,y)$ such that

$$\frac{\partial F}{\partial y}(x,y) = f(x,y)$$

and then

$$\int_{b(x)}^{t(x)} f(x,y)\,dy = F\bigl(x,t(x)\bigr) - F\bigl(x,b(x)\bigr)$$

The book calls this *partial integration with respect to y*.

Notice as promised that $\int_{b(x)}^{t(x)} f(x,y)\,dy$ is a function of x alone so the outer integral in the iterated integral is a first year calculus definite integral.

Fubini's Theorem: If f is continuous on the rectangle $R = \bigl\{(x,y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\bigr\}$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Note that in Fubini's Theorem you have your choice of in which order to do the integration. The two choices are equal but not necessarily equally easy to do. For example

$$\int_1^2 \int_0^\pi y\sin(xy)\,dx\,dy$$

is a straightforward substitution whereas

$$\int_0^\pi \int_1^2 y\sin(xy)\,dy\,dx$$

is a parts and then a substitution.