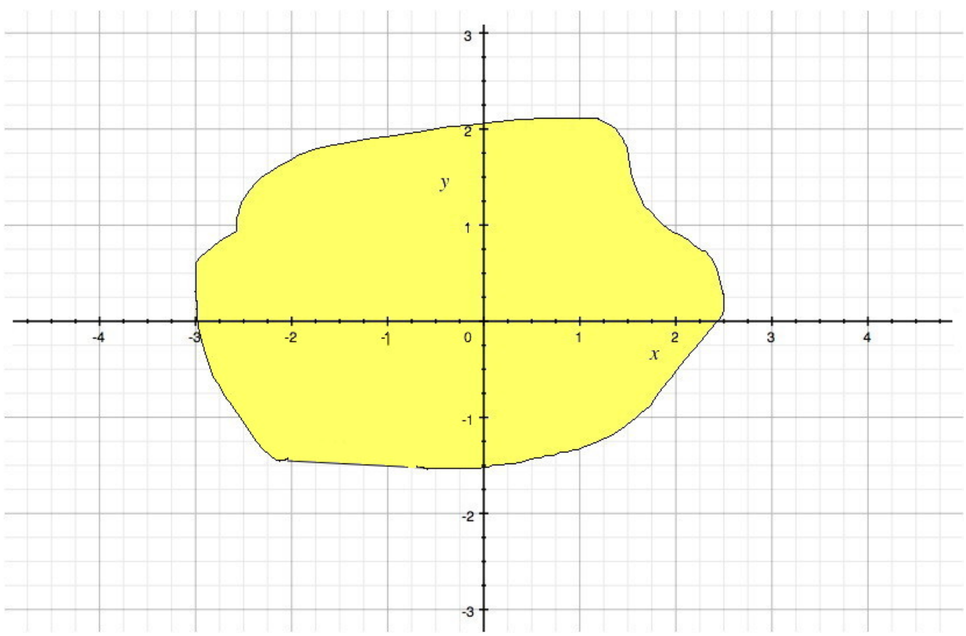
1. Regions again

The book talks about Type I and Type II regions as explains how to set up double integrals as iterated integrals for regions of either type. Using the sum theorem we discussed last time you can also do a region which is a union of regions of different types.

In principal you can set up any double integral as an iterated integral with $dA = dx\,dy$ or with $dA = dy\,dx$. It all depends on how good you are at answering a pair of questions.

Consider a bounded region R, for example the yellow region pictured here:

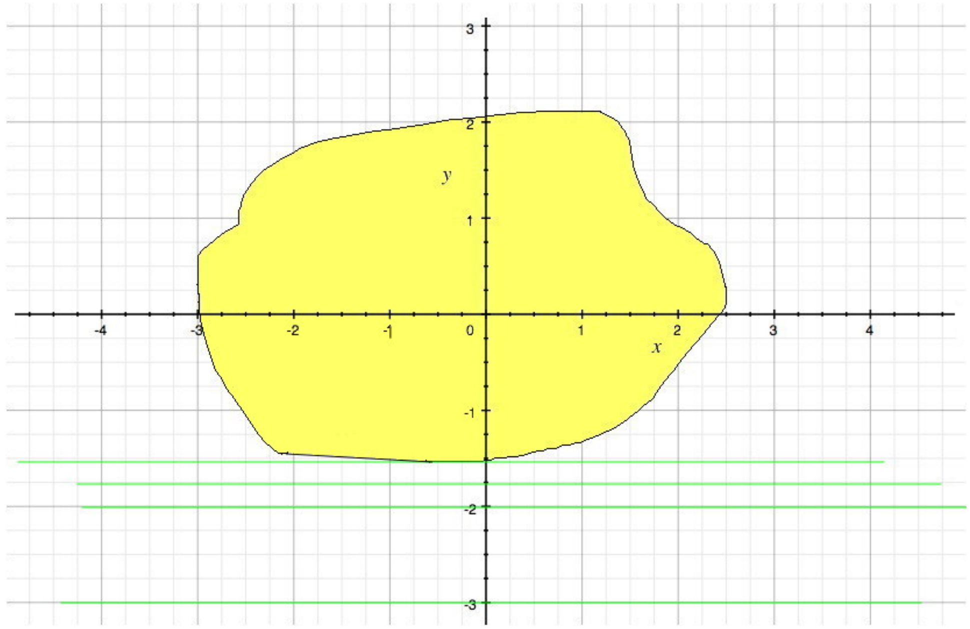


1.1. $dA = dx\,dy$.

Suppose you decide you want to do

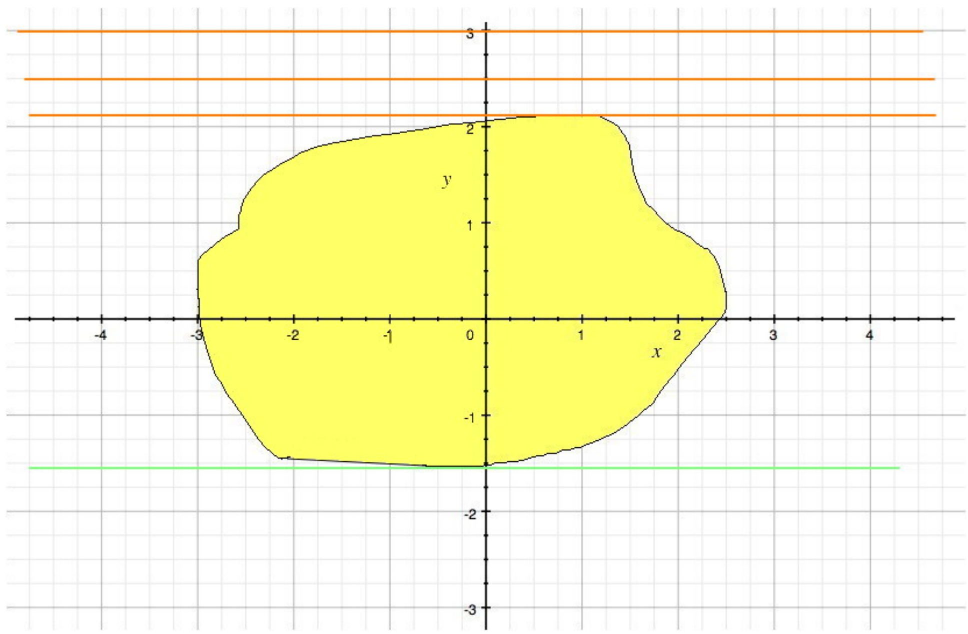
$$\iint_R f(x,y)\,dA = \int_?^? \int_?^? f(x,y)\,dx\,dy$$

The first question to answer is to get the y-limits (or the limits on the outside integral) right. Start with the lower limit. Ask yourself, what is the line $y = c$ with biggest c that just touches R from underneath?



The green horizontal line at $y = -3$ is too low as are the lines at $y = -2$ and $y = -1.75$. The line at $y = -1.5$ just touches the region so this is the lower limit.

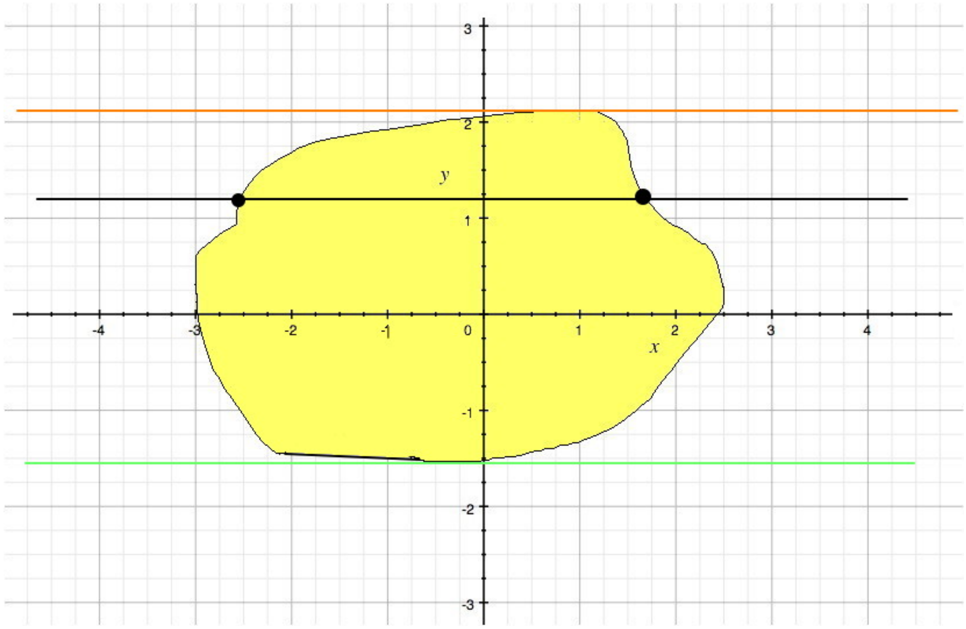
To get the upper limit, repeat the process except now you need to find the smallest a so the $y = d$ just touches R from above.



The orange horizontal line at $y = 3$ is too high as is the line at $y = 2.5$. The line at $y \approx 2.1$ just touches the region so this is the upper limit.

$$\iint_R f(x, y)\, dA = \int_{-1.5}^{2.1} \int_{?}^{?} f(x, y)\, dx\, dy$$

Now it is time to turn to the inside limits. These will be functions of y defined by looking at the region again. Of course these functions can be constant (as when you are integrating over a rectangle) but they may NOT involve the variable x.



The black horizontal line is supposed to represent an arbitrary horizontal line between the upper and lower y-limits. What you need is a formula for the left-hand dot in terms of y, $x = \ell(y)$, where the dot is where the horizontal line enters the region. You also need a formula for the right-hand dot in terms of y, $x = r(y)$ where the dot is where the horizontal line leaves the region.

To get the formula below it is vital that the entire interval between $\ell(x)$ and $r(x)$ lie in the region! (As it does in the picture.)

$$\iint_R f(x, y)\, dA = \int_{-1.5}^{2.1} \int_{\ell(y)}^{r(y)} f(x, y)\, dx\, dy$$

aaaa With just pictures it seems daunting to actually find these numbers and functions but in practice (and WITH practice) you will get quite good at it.

A type II region is just one where these questions are really easy to answer. Recall that a type II region is described as

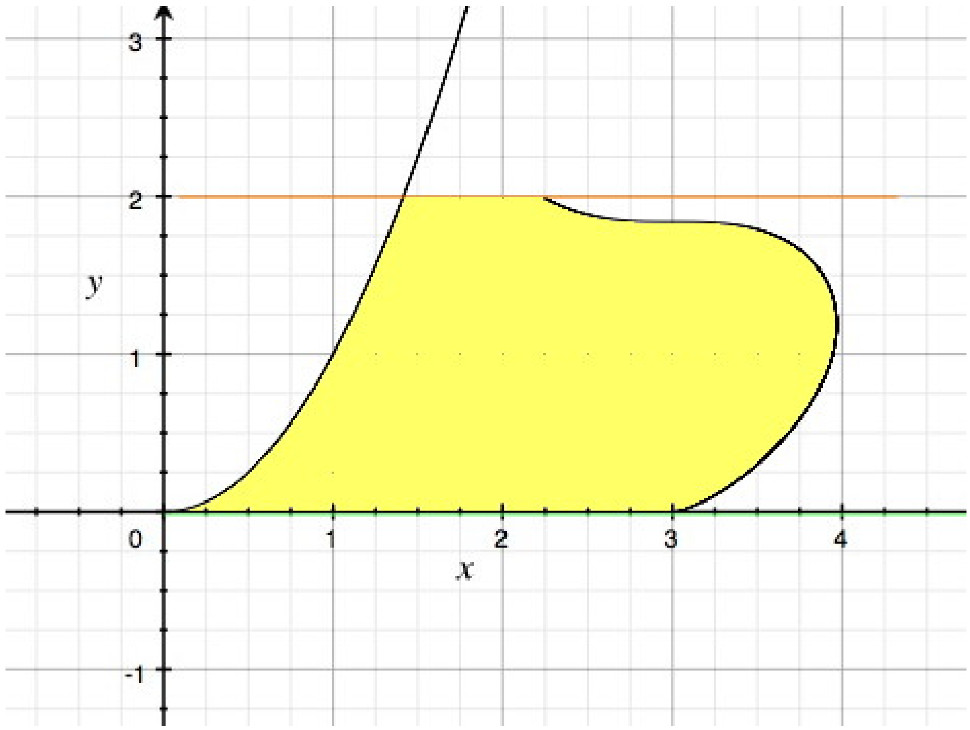
$$D = \big\{(x, y) \bigm| c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\big\}$$

so

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Here is a picture of

$$D = \left\{ (x,y) \;\middle|\; 0 \leqslant y \leqslant 2, \sqrt{y} \leqslant x \leqslant 3 + \sqrt[3]{\frac{y}{1.17}\sin\left(\frac{2y}{1.17}\right)} \;\right\}$$



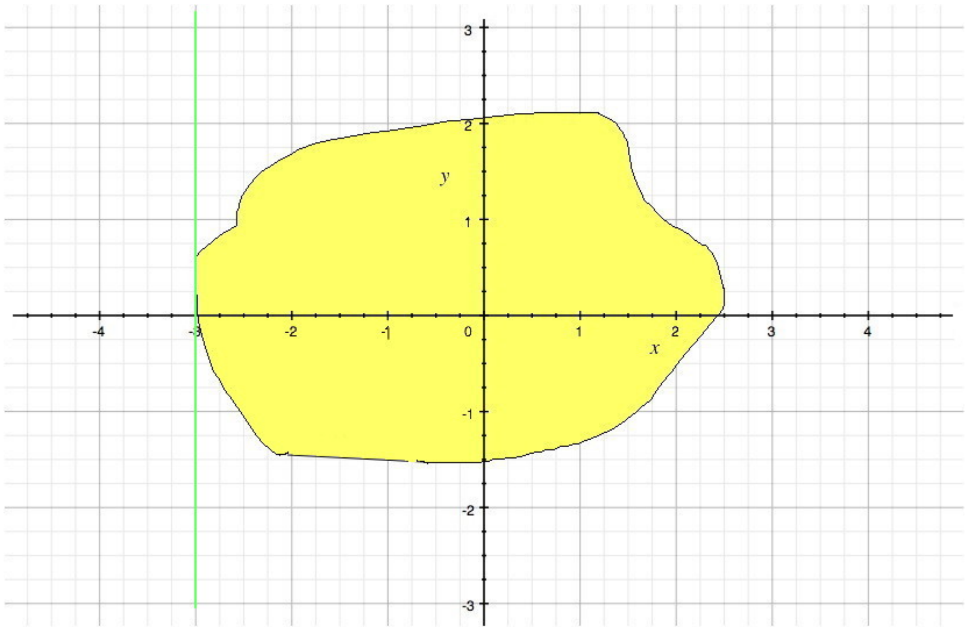
so

$$\iint_D f(x,y)\,dA = \int_0^2 \int_{\sqrt{y}}^{3+\sqrt[3]{\frac{y}{1.17}\sin\left(\frac{2y}{1.17}\right)}} f(x,y)\,dx\,dy$$

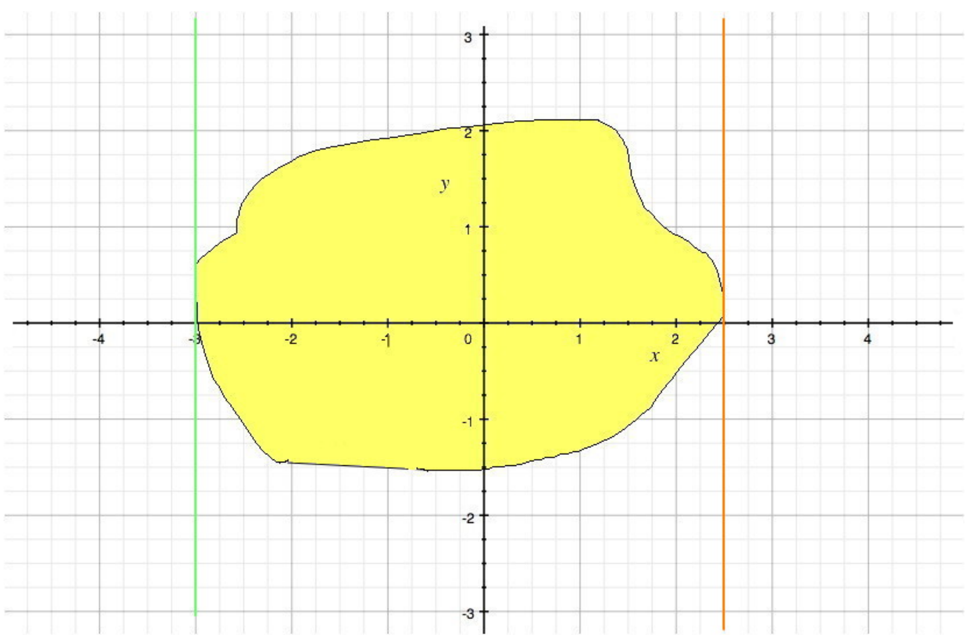
1.2. $dA = dy\,dx$. We want to understand the limits in the evaluation

$$\iint_R f(x,y)\,dA = \int_?^? \int_?^? f(x,y)\,dy\,dx$$

This is the same idea as in §1.1 except we start with locating x values. First locate the line closest to the origin so that the region lies to its right. It is the green line in the next picture.



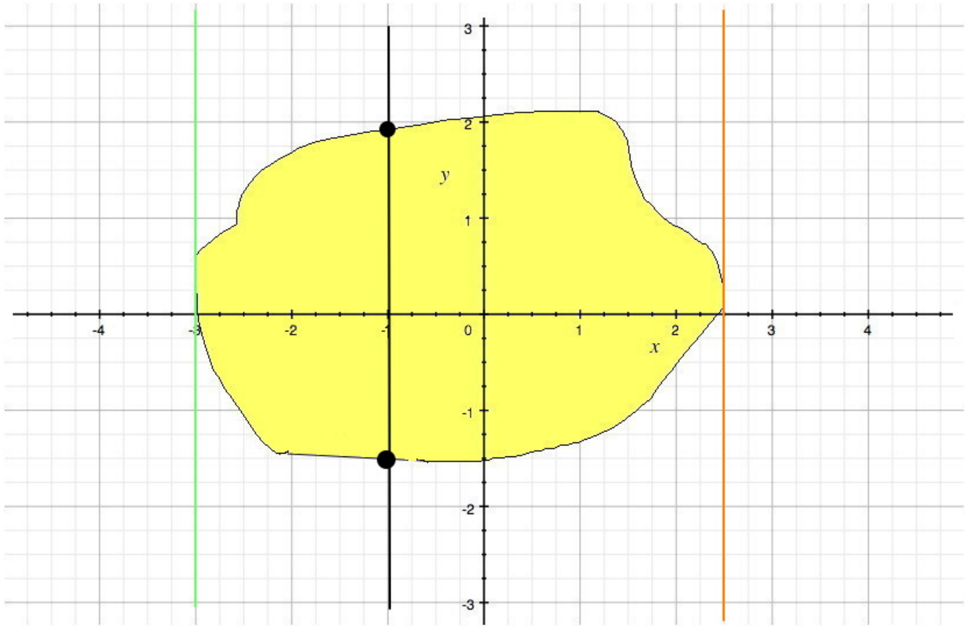
and if it is at $x = a$ the bottom limit on the outside integral is a. The vertical line which is closest to the origin for which the region lies to the left of the line. In the next picture this is the orange line and if this line is $x = b$, the upper limit in the outside integral is b.



and

$$\iint_R f(x,y)\,dA = \int_a^b \int_?^? f(x,y)\,dy\,dx$$

Then draw a "generic" vertical line between the green and orange lines to figure out the two functions.

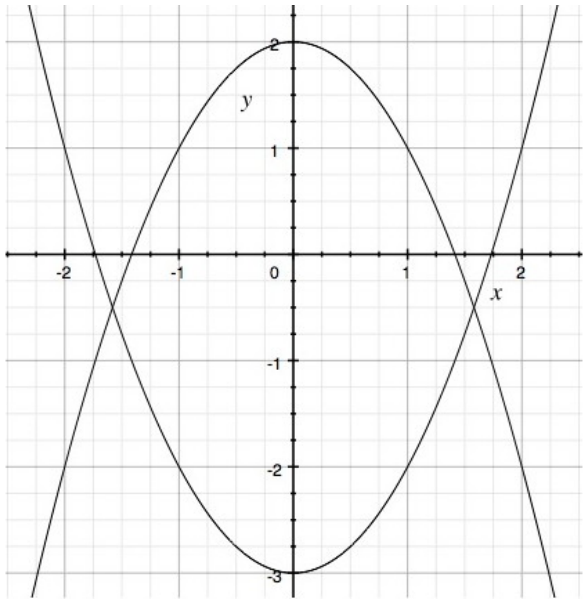


In this picture, the bottom black dot has y-coordinate $b(x)$ and the top black dot has y-coordinate $t(x)$.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{b(x)}^{t(x)} f(x,y)\,dy\,dx$$

This is an example of what the book calls a Type I region although the region is the same as in §1.1 where we saw it was a Type II region.

Example: Set up an iterated integral to evaluate $\displaystyle\iint_R f(x,y)\,dA$ where R is the region below $y = 2 - x^2$ and above $y = x^2 - 3$. Let $f(x,y) = x$. To do this as $dy\,dx$ we first need to find the appropriate vertical lines. They are the two vertical through the two intersection points.



The intersections are given by $2 - x^2 = x^2 - 3$ or $2x^2 = 5$ or $x = \pm\sqrt{\dfrac{5}{2}}$. At a generic vertical line, the lower function is $y = x^2 - 3$ and the upper function is $2 - x^2$. Hence

$$\iint_R x\,dA = \int_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} \int_{x^2-3}^{2-x^2} x\,dy\,dx$$

To actually do the integral

$$\int_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} \int_{x^2-3}^{2-x^2} x\,dy\,dx = \int \int_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} xy\Big|_{y=x^2-3}^{y=2-x^2} dx = \int_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} x\big((2-x^2)-(x^2-3)\big)\,dx =$$

$$\int_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} x\big(5-2x^2\big)\,dx = \int_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} 5x - 2x^3\,dx = \left(\frac{5x^2}{2} - \frac{2x^4}{4}\right)\Bigg|_{-\sqrt{\frac{5}{2}}}^{\sqrt{\frac{5}{2}}} = 0$$

since the powers of x are even and the limits are plus and minus of each other.

Notice that in this example of a Type I region the vertical lines are very short. In fact they are points.

You can do this example as $dx\,dy$. This time the lower line is $y = -3$ and the upper line is $y = 2$. Down near $y = -3$, the left hand function is $-\sqrt{y+3}$ and the right hand function is $\sqrt{y+3}$. Up near $y = 2$, the left hand function is $-\sqrt{2-y}$ and the right hand function is $\sqrt{2-y}$. This change of functions occurs along the line $y = -\frac{1}{2}$ so

$$\iint_R x\,dA = \int_{-3}^{-\frac{1}{2}} \int_{-\sqrt{y+3}}^{\sqrt{y+3}} x\,dx\,dy + \int_{-\frac{1}{2}}^{2} \int_{-\sqrt{2-y}}^{\sqrt{2-y}} x\,dx\,dy =$$

$$\int_{-3}^{-\frac{1}{2}} \frac{x^2}{2}\Big|_{-\sqrt{y+3}}^{\sqrt{y+3}} dy + \int_{-\frac{1}{2}}^{2} \frac{x^2}{2}\Big|_{-\sqrt{2-y}}^{\sqrt{2-y}} dy = \int_{-3}^{-\frac{1}{2}} 0\,dy + \int_{-\frac{1}{2}}^{2} 0\,dy = 0$$

Hence the region in this example is a type I region and the union of two typeII regions.

2. The reverse problem

It is also important to be able to go from an iterated integral to a double integral over a region,

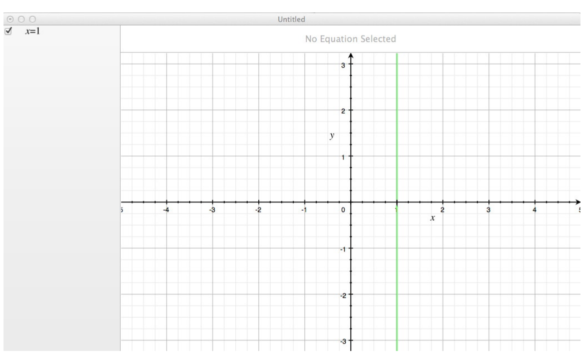
$$\int_a^b \int_{b(x)}^{t(x)} f(x,y)\,dy\,dx = \iint_R f(x,y)\,dA \quad \text{or} \quad \int_c^d \int_{\ell(y)}^{r(y)} f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$$

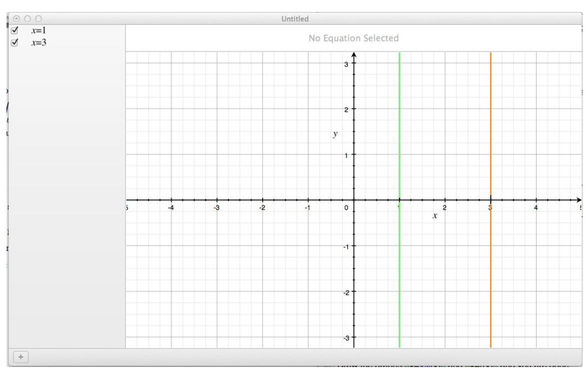
Since the function $f(x,y)$ does not change, the problem is to draw R given the iterated integral data. But if you remember how to go from the region to the iterated integral, this is an easy problem.

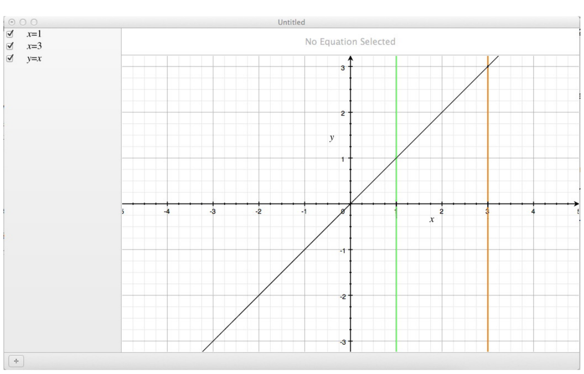
Suppose we want to describe R using the first iterated integral above. Look at the outside integral, $\displaystyle\int_a^b \cdots\,dx$. Draw the lines $x = a$ and $x = b$. Then look at the inner integral, $\displaystyle\int_{b(x)}^{t(x)} f(x,y)\,dy$. Draw the graphs $y = b(x)$ and $y = t(x)$ and you are done.

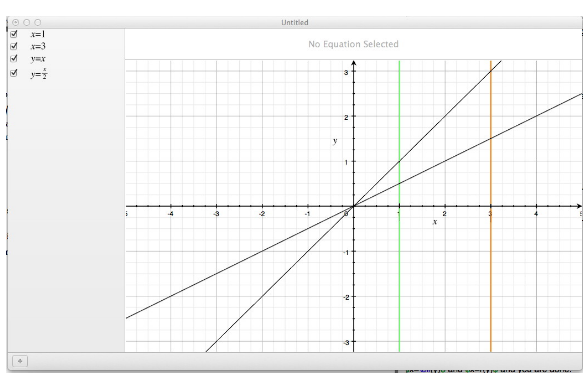
For the second iterated integral above look at the outside integral, $\displaystyle\int_c^d \cdots\,dx$. Draw the lines $y = c$ and $y = d$. Then look at the inner integral, $\displaystyle\int_{\ell(y)}^{r(y)} f(x,y)\,dx$. Draw the graphs $x = \ell(y)$ and $x = r(y)$ and you are done.

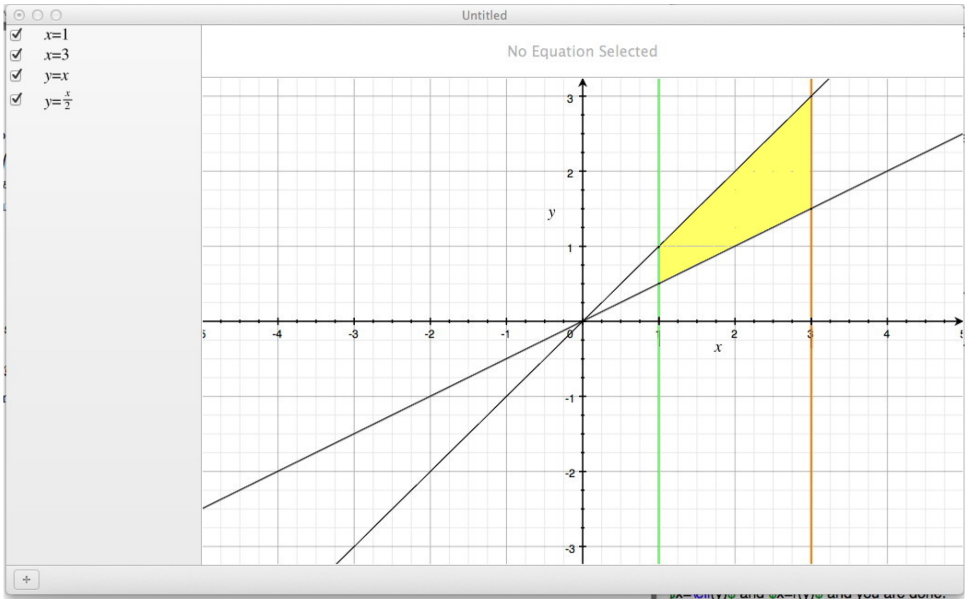
Example: $\displaystyle\int_1^3 \int_{x/2}^{x} \cdots\,dy\,dx$. Here is a sequence of shots from my graphing prorgam.



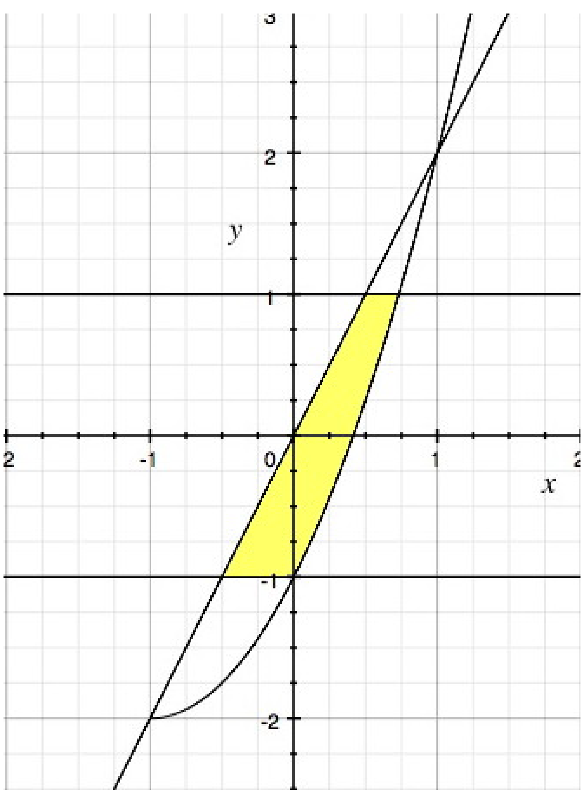








Example: $\displaystyle\int_{-1}^{1}\int_{-1+\sqrt{y+2}}^{y/2} \cdots\, dx\, dy.$



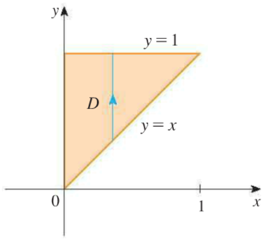


FIGURE 15
D as a type I region

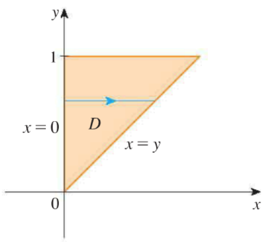


FIGURE 16
D as a type II region

V EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\,dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\,dy$ is not an elementary function. (See the end of Section 7.5.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using $\boxed{3}$ backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

where $$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

This enables us to use $\boxed{5}$ to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}
\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\
&= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \Big[x \sin(y^2)\Big]_{x=0}^{x=y} dy \\
&= \int_0^1 y \sin(y^2)\,dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1 = \tfrac{1}{2}(1 - \cos 1)
\end{aligned}$$