Example. Look at some sort of twisted cubic $\mathbf{r}(t) = \langle t^2 - t, 1, t^3 \rangle$. Find the points on the curve where the curvature vanishes.

$\mathbf{r}'(t) = \langle 2t-1, 0, 3t^2 \rangle \qquad |\mathbf{r}'(t)| = \sqrt{(2t-1)^2 + 9t^4}$

$\mathbf{r}''(t) = \langle 2, 0, 6t \rangle$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t-1 & 0 & 3t^2 \\ 2 & 0 & 6t \end{vmatrix} = \begin{vmatrix} 0 & 3t^2 \\ 0 & 6t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2t-1 & 3t^2 \\ 2 & 6t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2t-1 & 0 \\ 2 & 0 \end{vmatrix}\mathbf{k} = \langle 0, -6t^2+6, 0 \rangle$$

The curvature is

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{\left(-6t^2+6\right)^2}}{\left(\sqrt{(2t-1)^2+9t^4}\right)^3}$$

Hence the curvature vanishes if and only if $-6t^2 + 6 = 0$ or $t = \pm 1$.

Hence the curvature is 0 at the two points $\langle 0, 1, 1 \rangle$ and $\langle 0, 1, -1 \rangle$.

Example. What is the torsion of $\mathbf{r}(t) = \langle t^2 - t, 1, t^3 \rangle$?

$\mathbf{r}'''(t) = \langle 0, 0, 6 \rangle$

$$\left(\mathbf{r}'(t)\times\mathbf{r}''(t)\right) \bullet \mathbf{r}'''(t) = \left\langle 0, -6t^2+6, 0 \right\rangle \bullet \langle 0, 0, 6 \rangle = 0$$

Notice this confirms are previous remark that a curve is planar if and only if it torsion is 0 since the curve lies in the plane $y = 1$.

Notice as long as we're here,

$$\mathbf{B} = \frac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|} = \frac{\langle 0, -6t^2+6, 0 \rangle}{\sqrt{(-6t^2+6)^2}} = \begin{cases} \langle 0, 1, 0 \rangle & t^2 < 1 \\ \langle 0, -1, 0 \rangle & t^2 > 1 \end{cases}$$

Then

$$\mathbf{N} = \mathbf{B}\times\mathbf{T}$$

This points in the same direction as

$$\langle 0, \pm 1, 0 \rangle \times \left\langle 2t-1, 0, 3t^2 \right\rangle =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \pm 1 & 0 \\ 2t-1 & 0 & 3t^2 \end{vmatrix} = \begin{vmatrix} \pm 1 & 0 \\ 0 & 3t^2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ 2t-1 & 3t^2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & \pm 1 \\ 2t-1 & 0 \end{vmatrix}\mathbf{k} =$$

$$\left\langle \pm 3t^2, 0, \pm(2t-1) \right\rangle$$

Hence

$$\mathbf{N} = \frac{1}{\sqrt{9t^4 + (2t-1)^2}} \left\langle \pm 3t^2, 0, \pm(2t-1) \right\rangle = \begin{cases} \dfrac{\langle 3t^2, 0, 2t-1 \rangle}{\sqrt{9t^4+(2t-1)^2}} & t^2 < 1 \\ \dfrac{-\langle 3t^2, 0, 2t-1 \rangle}{\sqrt{9t^4+(2t-1)^2}} & t^2 < 1 \end{cases}$$

Example. Look at this quartic cubic $\mathbf{r}(t) = \langle t^3, t, t^4 \rangle$. Find all the points on the cubic where the osculating plane is parallel to $-12x + 24y + 3z = 5$.

$\mathbf{r}'(t) = \langle 3t^2, 1, 4t^3 \rangle$

$\mathbf{r}''(t) = \langle 6t, 0, 12t^2 \rangle$

A normal to the osculating plane is $\mathbf{B}$ which is parallel to

$$\mathbf{n} = \mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3t^2 & 1 & 4t^3 \\ 6t & 0 & 12t^2 \end{vmatrix} = \begin{vmatrix} 1 & 4t^3 \\ 0 & 12t^2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3t^2 & 4t^3 \\ 6t & 12t^2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3t^2 & 1 \\ 6t & 0 \end{vmatrix} \mathbf{k} = \langle 12t^2, -12t^4, -6t \rangle$$

We certainly should not waste time normalizing $\mathbf{n}$. However $\mathbf{n} = -6t\,\langle -2t, 2t^3, 1 \rangle$ so a normal to the osculating plane is $\mathbf{n}_1 = \langle -2t, 2t^3, 1 \rangle$ and this is the normal vector we will use.

A normal to $-12x + 24y + 3z = 5$ is $\mathbf{n}_2 = \langle -12, 24, 3 \rangle$ and the only way $\mathbf{n}_1$ and $\mathbf{n}_2$ can be parallel is if $\mathbf{n}_2 = 3\mathbf{n}_1$. Hence $-6t = -12$ and $6t^3 = 24$. The first equation forces $t = 2$ but $t = 2$ does not satisfy the second equation. Hence there is no point on the curve where the osculating plane is parallel to $-12x + 24y + 3z = 5$.

If instead we wanted a plane parallel to $-12x + 48y + 3z = 5$ then the only possible solution is still $t = 2$ since the first and third coordinates of our normals to the planes are the same. But now $6t^3 = 48$ is satisfied when $t = 2$.

The point on the curve is $\mathbf{r}(2) = \langle 8, 2, 16 \rangle$.

An osculating plane equation is

$$\langle -12, 24, 3 \rangle \bullet \langle x, y, z \rangle = \langle -12, 24, 3 \rangle \bullet \langle 8, 2, 16 \rangle = -96 + 48 + 48 = 0$$

or

$$-4x + 8y + z = 0$$

Example. Find the cosine of the angle of intersection between the curves

(1) $\mathbf{r}_1(t) = \langle t + 1, t^3 - t, t^2 - 1 \rangle$
(2) $\mathbf{r}_2(t) = \langle t, t^2 - 1, -t^2 \rangle$

First we need to compute the point(s) of intersection. Switch the variable in one of the equations, say $\mathbf{r}_2(s) = \langle s, s^2 - 1, -s^2 \rangle$ and solve

$$\left\langle t + 1, t^3 - t, t^2 - 1 \right\rangle = \left\langle s, s^2 - 1, -s^2 \right\rangle$$

Hence $s = t + 1$ and then $s^2 = t^2 + 2t + 1$. Then $t^2 - 1 = -s^2 = -(t^2 + 2t + 1)$ so $2t^2 + 2t = 0$ so $t = 0$ and $t = -1$. If $t = 0$, $s = 1$ and if $t = -1$, $s = 0$.

When $t = 0$ and $s = 1$ the point is $(1, 0, -1)$. When $t = -1$ and $s = 0$ we get two different points $(0, 0, 0)$ and $(0, -1, 0)$ so there is only one point of intersection.

A tangent vector to each curve is $\mathbf{r}_1'(t) = \langle 1, 3t^2 - 1, 2t \rangle$ and $\mathbf{r}_2'(s) = \langle 1, 2s, -2s \rangle$. At the intersection we have $\mathbf{r}_1'(0) = \langle 1, -1, 0 \rangle$ and $\mathbf{r}_2'(1) = \langle 1, 2, -2 \rangle$.

$$\cos(\theta) = \frac{\langle 1, -1, 0 \rangle \bullet \langle 1, 2, -2 \rangle}{|\,\langle 1, -1, 0 \rangle\,| \cdot |\,\langle 1, 2, -2 \rangle\,|} = \frac{-1}{\sqrt{2} \cdot \sqrt{9}} = \frac{-1}{3\sqrt{2}}$$