1. Vectors

- Dot product: $\langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$.
  - Angle between two vectors: $\cos(\theta) = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}$
  - Perpendicular and orthogonal: $\mathbf{a}\bullet\mathbf{b} = 0$
  - Projection of $\mathbf{b}$ onto $\mathbf{a}$: $\dfrac{\mathbf{a}\bullet\mathbf{b}}{\mathbf{a}\bullet\mathbf{a}}\mathbf{a}$
  - Work: Force $\bullet$ Displacement
  - Rules: $\mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2$; $\mathbf{a}\bullet\mathbf{b} = \mathbf{b}\bullet\mathbf{a}$; $\mathbf{a}\bullet(\mathbf{b}+\mathbf{c}) = \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{c}$;
    $(c\mathbf{a})\bullet\mathbf{b} = \mathbf{a}\bullet(c\mathbf{b}) = c(\mathbf{a}\bullet\mathbf{b})$; $\mathbf{a}\bullet\mathbf{0} = 0$.

- Cross product: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$
  - Vector orthogonal to two vectors: $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and to $\mathbf{b}$.
  - Area of parallelogram: $A = |\mathbf{a}\times\mathbf{b}|$
  - Angle between two vectors: $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin(\theta)$
  - Rules: $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$; $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$;
    $(c\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(c\mathbf{b}) = c(\mathbf{a}\times\mathbf{b})$;
  - $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{b} & \mathbf{c} \\ \mathbf{a}\bullet\mathbf{b} & \mathbf{a}\bullet\mathbf{c} \end{vmatrix}$ & $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{b} & \mathbf{a} \\ \mathbf{b}\bullet\mathbf{c} & \mathbf{a}\bullet\mathbf{c} \end{vmatrix}$

- Triple scalar product: $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$
  - To calculate: $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
  - Volume of parallelepiped: $|\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})|$
  - Rule: $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}$.

- Equations of a line
  - Vector equation: $t\langle v_1, v_2, v_3\rangle + \langle c_1, c_2, c_3\rangle$
  - Symmetric equation: $\dfrac{x-c_1}{v_1} = \dfrac{y-c_2}{v_2} = \dfrac{z-c_3}{v_3}$ with adjustments when $v_i = 0$.

- Equation of a plane: normal vector $\mathbf{N} = \langle a, b, c\rangle$ and point $\langle p_1, p_2, p_3\rangle$ gives the equation $\mathbf{N}\bullet\langle x, y, z\rangle = \mathbf{N}\bullet\langle p_1, p_2, p_3\rangle$.

2. CURVES

2.1. **Parametrization.** Vector valued function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. A parametrization is smooth provided $\mathbf{r}'(t) \neq \mathbf{0}$. A smooth parametrization orients the curve.

2.2. **Arc Length.** Arc length between points $\mathbf{p}_1$ and $\mathbf{p}_2$: find t_1 and t_2 such that $\mathbf{r}(t_i) = \mathbf{p}_i$ and if the parametrization is smooth, the arc length is $\displaystyle\int_{t_1}^{t_2} |\mathbf{r}'(t)|\, dt$ provided $t_1 < t_2$.

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

2.3. **Tangent vector(s).** A tangent vector to the curve at t is $\mathbf{r}'(t)$. At a point $\mathbf{p}$ find c such that $\mathbf{r}(c) = \mathbf{p}$ and use $\mathbf{r}'(c)$.

2.4. **Velocity and acceleration.** Velocity $= \mathbf{r}'(t)$, speed $= |\mathbf{r}'(t)|$. Acceleration $= \mathbf{r}''(t)$.

2.5. **Frenet-Serret frame.** Unit tangent $\mathbf{T}(t)$, unit normal $\mathbf{N}(t)$, and unit binormal, $\mathbf{B}(t)$. These three unit vectors are mutually orthogonal and $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ is a right-handed frame.

$$\mathbf{r}''(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t) \text{ where } a_{\mathbf{T}}(t) = \frac{d\,|\mathbf{r}'(t)|}{dt} \text{ and } a_{\mathbf{N}}(t) = \kappa(t)\,|\mathbf{r}'(t)|^2.$$

If you actually want to compute $a_{\mathbf{T}}(t)$ use $a_{\mathbf{T}}(t) = \dfrac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$.

If you actually want to compute $\kappa(t)$ use $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

If you actually want to compute $a_{\mathbf{N}}(t)$ use $a_{\mathbf{N}}(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$.

If you actually want to compute the vectors in the Frenet-Serret frame

- $\mathbf{T}(t)$: take $\mathbf{r}'(t)$ and make it unit length.
- $\mathbf{B}(t)$: take $\mathbf{r}'(t) \times \mathbf{r}''(t)$ and make it unit length.
- $\mathbf{N} = \mathbf{B} \times \mathbf{T}$: take $\big(\mathbf{r}'(t) \times \mathbf{r}''(t)\big) \times \mathbf{r}'(t) = \begin{vmatrix} \mathbf{r}''(t) & \mathbf{r}'(t) \\ \mathbf{r}''(t) \bullet \mathbf{r}'(t) & \mathbf{r}'(t) \bullet \mathbf{r}'(t) \end{vmatrix}$ and make it unit length.

If you want any of these quantities at a single value of t, compute $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ at the point and then work with vectors of numbers.

The *osculating plane* to a curve at a point is the plane spanned by $\mathbf{T}(t)$ and $\mathbf{N}(t)$. A normal vector to this plane is $\mathbf{B}(t)$. If you actually want to find an equation for the osculating plane, use $\mathbf{r}' \times \mathbf{r}''$ for the normal vector.

The *normal plane* to a curve at a point is the plane with normal vector $\mathbf{T}$ through the point.

2.6. **Antiderivatives.** Find the equation of a curve given an equation for velocity, $\mathbf{v}(t)$. $\displaystyle\int \mathbf{v}(t)\, dt$ is the family of curves with that velocity. One can also compute the velocity from the acceleration up to a constant vector.