1. Review

1.1. Surface integrals. Let T be a surface in $\mathbb{R}^3$. Let $f\colon T \to \mathbb{R}$ be a function defined on T. Define

$$\iint_T f\,dS = \lim_{mesh(\mathcal{P})\to 0} \sum_{\mathcal{P}} f(\mathbf{p}_i)\text{Area}(T_i)$$

as a limit of Riemann sums over sampled-partitions. A *sampled-partition of* T, $\mathcal{P}$, is a division of the surface T into pieces, T_i, followed by a choice of *sample point*, $\mathbf{p}_i$, in each T_i. By a division of a surface into pieces we mean $\cup_i T_i = T$ and $T_i \cap T_j$ lies in the boundary of each piece.

The associated Riemann sum is $\sum_i f(\mathbf{p}_i)\text{Area}(T_i)$. The *mesh* of the sampled-partition is $\leq \epsilon$ provided every T_i is contained in the ball of radius ϵ centered at $\mathbf{p}_i$.

If f is continuous, $\displaystyle\iint_T f\,dS$ is well-defined, and hence a number.

1.2. Surface integrals as double integrals. Parametrize S as $\mathbf{r}(u,w)\colon D \to \mathbb{R}^3$ where D is some bounded region in the plane. We can evaluate f at points in S as $f\big(\mathbf{r}(u,w)\big)$. The remaining issue is to figure out $\text{Area}(T_i)$.

At a point $\mathbf{r}(u_0,w_0)$ we can look at the tangent plane to S at the point. It is

$$\mathbf{r}(u_0,w_0) + r\mathbf{r}_u(u_0,w_0) + t\mathbf{r}_w(u_0,w_0)$$

In the tangent plane we have the parallelogram with vertices $\mathbf{r}(u_0,w_0)$, $\mathbf{r}(u_0,w_0)+\mathbf{r}_u(u_0,w_0)du$, $\mathbf{r}(u_0,w_0)+\mathbf{r}_w(u_0,w_0)dw$ and $\mathbf{r}(u_0,w_0)+\mathbf{r}_u(u_0,w_0)du+\mathbf{r}_w(u_0,w_0)dw$. The area of this parallelogram is $|\mathbf{r}_u(u_0,w_0)\times\mathbf{r}_w(u_0,w_0)|\,du\,dw = |\mathbf{r}_u(u_0,w_0)\times\mathbf{r}_w(u_0,w_0)|\,dA$ and we use this as the approximation to $\text{Area}(T_i)$.

$$(*) \qquad \iint_T f\,dS = \iint_D f\big(\mathbf{r}(u,w)\big)\,|\mathbf{r}_u\times\mathbf{r}_w|\,dA$$

2. Flux integrals

One important source of functions to integrate over surfaces comes using vector fields and the dot product. Let T be an *oriented surface*. Given any vector field $\mathbf{F}$ defined on T, define

$$\iint_T \mathbf{F}\bullet d\mathbf{S} = \iint_T \mathbf{F}\bullet\mathbf{N}\,dS$$

where $\mathbf{N}$ is the unit normal field to T, *which is why T has to be oriented.*

Since $\mathbf{F}$ and $\mathbf{N}$ are both vector fields defined on T, $\mathbf{F}\bullet\mathbf{N}$ is a function defined on T and hence the right hand surface integral is defined.

Just as in the line integral case, the flux integral looks like it is going to be the most difficult of the surface integrals, but in fact it is the easiest. First recall,

$$\mathbf{N} = \frac{\mathbf{r}_u\times\mathbf{r}_w}{|\mathbf{r}_u\times\mathbf{r}_w|}$$

so

$$\iint_T \mathbf{F}\bullet d\mathbf{S} = \iint_D \mathbf{F}\big(\mathbf{r}(u,w)\big)\bullet\frac{\mathbf{r}_u\times\mathbf{r}_w}{|\mathbf{r}_u\times\mathbf{r}_w|}\,|\mathbf{r}_u\times\mathbf{r}_w|\,dA = \iint_D \mathbf{F}\big(\mathbf{r}(u,w)\big)\bullet(\mathbf{r}_u\times\mathbf{r}_w)\,dA$$

3. Orientation

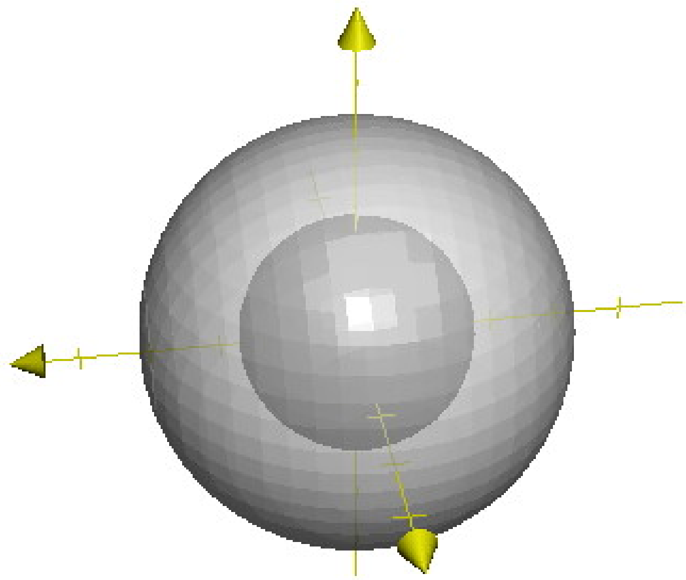
An *orientation* for a surface is a continuous choice of unit normal vector for the surface. Every surface which can be parametrized has two choices of a unit normal vector at every point, hence *if there is an orientation*, there are at least two.

The boundary of a solid E in three space has orientations. By ∂E we mean the boundary of E with the orientation for which the normal vector points out of E.

Remark Also just as in the line integral case, the surface integral does not depend on the orientation, but $\mathbf{F} \bullet \mathbf{N}$ changes sign if the orientation does. Hence the value of the flux integral depends on the orientation and changes sign if the orientation changes.

Below we will have occasion to do flux integrals over surfaces that some in several disjoint pieces. Then we will have to orient each piece and if we switch the orientation on just one piece then only the sign of the answer for that piece changes. In this case, if there are k pieces, there are 2^k possible orientations.

Consider the case in which E is the region between two shpheres.



This is an example in which ∂E is in two pieces, the inner sphere and the outer sphere. When we write ∂E we mean the boundary plus an orientation on it. The orientation on the outer sphere points outward but the orientation on the inner sphere points inward. *They both point out of E.*

As surfaces, each sphere has two orientations for a total of 4 altogether on the union of the two.

Once you have parametrized the surface, you have a natural choice for normal vector, the cross product of the two partial derivatives. If the surface is orientable, this works. If the surface is not orientable, nothing works.

We saw the Möbius strip as an example of a non-orientable surface. In general, any surface without boundary is orientable. Additionally, any subset of an oriented surface is oriented.

Hence level sets are orientable and surfaces of revolution are orientable. Examples include graphs, cylinders, cones, ...

4. Examples

Example: Find $\iint\limits_T f\,dS$ where T is the surface consisting of the part of the plane $2x+3y+z=6$ lying in the first octant and $f(x,y,z)=x+y+z$.
Step 1: First parametrize the surface: $\mathbf{r}(x,y)=\langle x,y,6-2x-3y\rangle$; (x,y) in D where D is the triangle in the first quadrant of the xy plane below the line $2x+3y=6$.
Step 2: Calculate the normal field.

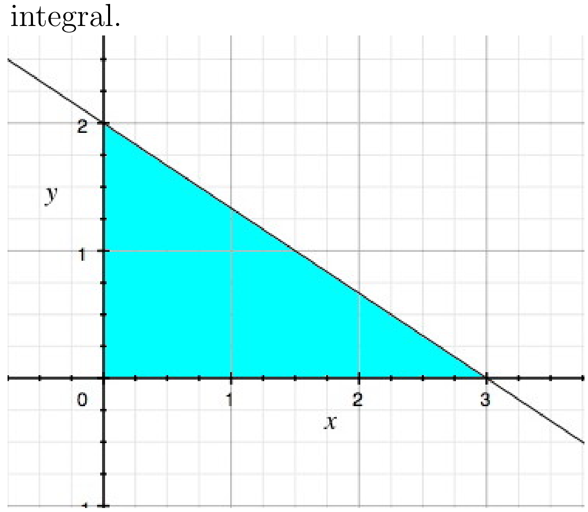
$$\mathbf{r}_x=\langle 1,0,-2\rangle$$
$$\mathbf{r}_y=\langle 1,0,-3\rangle$$
$$\mathbf{r}_x\times\mathbf{r}_y=\det\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&-2\\0&1&-3\end{vmatrix}=\langle 2,3,1\rangle$$

Step 3: Reduce to a double integral. This is not a flux integral so there is no need to worry about the orientation. However we need to compute the length of the normal field.

$$|\mathbf{r}_x\times\mathbf{r}_y|=\sqrt{4+9+1}=\sqrt{14}$$

$$\iint\limits_T f\,dS=\iint\limits_D (x+y+z)\sqrt{14}\,dA=\sqrt{14}\iint\limits_D (x+y+6-2x-3y)\,dA=\sqrt{14}\iint\limits_D (6-x-2y)\,dA$$

Step 4: Evaluate the double integral.



Draw a picture: $2x+3y=6$.

$$\sqrt{14}\int_0^3\int_0^{2-\frac{2x}{3}}(6-x-2y)\,dy\,dx=\sqrt{14}\int_0^3 6y-xy-y^2\Big|_0^{2-\frac{2x}{3}}\,dx=$$

$$\sqrt{14}\int_0^3\left((12-4x)-2x+\frac{2x^2}{3}-\left(4-\frac{8x}{3}+\frac{4x^2}{9}\right)\right)dx=\sqrt{14}\int_0^3\left(8-\frac{10x}{3}+\frac{2x^2}{9}\right)dx=$$

$$\sqrt{14}\left(8x-\frac{5x^2}{3}+\frac{2x^3}{27}\Big|_0^3\right)=\sqrt{14}(24-15+2)=11\sqrt{14}$$

You may certainly evaluate this double integral as $dx\,dy$.

Example: Find $\iint\limits_T \mathbf{F} \bullet d\mathbf{S}$ where T is the surface in the last example with downward normal and where $\mathbf{F} = \langle x, y, z\rangle$.

The region D and the calculation of $\mathbf{r}_x \times \mathbf{r}_y = \langle 2, 3, 1\rangle$ are the same as for the last example.
Step 3: Reduce to a double integral. This is a flux integral so we need to check the orientation. The normal vector at all points is $\langle 2, 3, 1\rangle$ which points up since the z coordinate is positive. We wanted downward so

$$\iint\limits_T \mathbf{F} \bullet d\mathbf{S} = -\iint\limits_D \langle x, y, z\rangle \bullet \langle 2, 3, 1\rangle\, dA = -\iint\limits_D (2x + 3y + 6 - 2x - 3y)\, dA = -\iint\limits_D 6\, dA$$

Step 4: Evaluate the double integral.

$$\iint\limits_D 6\, dA = 6 \cdot \frac{2 \cdot 3}{2} = 18$$

so

$$\iint\limits_T \mathbf{F} \bullet d\mathbf{S} = -18$$

Example: Let T be the part of $z = 4 - x^2 - y^2$ above the xy plane and let $\mathbf{F} = \langle y, x, z\rangle$. Let the normal field be upward.
Step 1: Parametrize T as $\mathbf{r}(x, y) = \langle x, y, 4 - x^2 - y^2\rangle$ for $D\colon x^2 + y^2 \leqslant 4$.
Step 2: Calculate the normal field.

$$\mathbf{r}_x(x, y) = \langle 1, 0, -2x\rangle$$
$$\mathbf{r}_y(x, y) = \langle 0, 1, -2y\rangle$$
$$\mathbf{r}_x \times \mathbf{r}_y = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1\rangle$$

Step 3: Reduce to a double integral. Check the orientation since this is a flux integral. When $(x, y) = (0, 0)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle 0, 0, 1\rangle$ which points up so the normal field is the one we wanted.

$$\mathbf{F} \bullet (\mathbf{r}_x \times \mathbf{r}_y) = \left\langle y, x, 4 - x^2 - y^2\right\rangle \bullet \langle 2x, 2y, 1\rangle = 2xy + 2xy + 4 - x^2 - y^2 = 4xy + 4 - x^2 - y^2$$

$$\iint\limits_T \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{x^2+y^2\leqslant 4} (4xy + 4 - x^2 - y^2) dA$$

Step 4: Evaluate the double integral. In this case, polar coordinates looks good.

$$\int_0^{2\pi}\int_0^2 (4r^2\cos(\theta)\sin(\theta) + 4 - r^2) r\, dr d\theta = \int_0^{2\pi}\int_0^2 (4r^3\cos(\theta)\sin(\theta) + 4r - r^3)\, dr d\theta =$$

$$\int_0^{2\pi} \left(r^4 \cos(\theta)\sin(\theta) + 2r - \frac{r^4}{4}\Bigg|_0^2 \right) d\theta = \int_0^{2\pi} \big(16\cos(\theta)\sin(\theta) + 4 - 4\big)\, d\theta = 8\sin^2(\theta)\Big|_0^{2\pi} = 0$$

Remark: In general, parametrize the surface and do the integral using that parametrization. After you do the calculation, check to see if you did it with the correct orientation or not. If you did it with the required orientation you are done. If you did it with the other orientation, then the correct answer is the negative of the one you worked out.

Example: Let T be the cone $z = \sqrt{x^2+y^2}$ below $z = 2$ together with the disk $x^2+y^2 \leqslant 4$, $z = 2$. Orient the surface so that the normal is outward. Integrate the field $\mathbf{F} = \langle x, y, 0\rangle$ over this surface.
Step 1: Parametrize T as $T_1 \cup T_2$ were T_1 is parametrized in spherical coordinates by $\phi = \frac{\pi}{4}$ and $\mathbf{r}_1(\rho,\theta) = \left\langle \rho\cos(\theta)\frac{\sqrt{2}}{2}, \rho\sin(\theta)\frac{\sqrt{2}}{2}, \rho\frac{\sqrt{2}}{2}\right\rangle = \frac{\sqrt{2}\rho}{2}\langle\cos(\theta),\sin(\theta),1\rangle$; $D_1\colon 0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \rho \leqslant 2\sqrt{2}$. The surface T_2 is parametrized by $\mathbf{r}_2(x,y) = \langle x, y, 2\rangle$; $D_2\colon x^2+y^2 \leqslant 4$.
Write the surface as a union of surfaces which you can parametrize and then $\iint\limits_T \mathbf{F}\bullet d\mathbf{S} = \iint\limits_{T_1} \mathbf{F}\bullet d\mathbf{S} + \iint\limits_{T_2} \mathbf{F}\bullet d\mathbf{S}$. Do each surface integral separately.

Step 2_1: Compute the normal field for T_1.

$$(\mathbf{r}_1)_\rho = \left\langle \frac{\sqrt{2}\cos(\theta)}{2}, \frac{\sqrt{2}\sin(\theta)}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$$(\mathbf{r}_1)_\theta = \left\langle -\frac{\sqrt{2}\rho\sin(\theta)}{2}, \frac{\sqrt{2}\rho\cos(\theta)}{2}, 0\right\rangle$$

$$\begin{aligned}(\mathbf{r}_1)_\rho \times (\mathbf{r}_1)_\theta &= \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{2}\cos(\theta)}{2} & \frac{\sqrt{2}\sin(\theta)}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}\rho\sin(\theta)}{2} & \frac{\sqrt{2}\rho\cos(\theta)}{2} & 0 \end{vmatrix} \\ &= \left\langle -\frac{\rho\cos(\theta)}{2}, -\frac{\rho\sin(\theta)}{2}, \rho\frac{\cos^2(\theta)+\sin^2(\theta)}{2}\right\rangle \\ &= \frac{\rho}{2}\langle -\cos(\theta), -\sin(\theta), 1\rangle\end{aligned}$$

Step 3_1: Reduce to a double integral.

$$\mathbf{F}\bullet\left((\mathbf{r}_1)_\rho \times (\mathbf{r}_1)_\theta\right) = \left\langle \rho\cos(\theta)\frac{\sqrt{2}}{2}, \rho\sin(\theta)\frac{\sqrt{2}}{2}, 0\right\rangle \bullet \frac{\rho}{2}\langle -\cos(\theta), -\sin(\theta), 1\rangle = -\left(\frac{\sqrt{2}}{4}\right)\rho^2$$

Check the orientation. We were asked to use the outward normal. If we use the point $\mathbf{r}_1(0,0)$ we get the cone point which is most likely singular, so let's use $\mathbf{r}_1(1,0) = \left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right\rangle$. Hence $(\mathbf{r}_1)_\rho \times (\mathbf{r}_1)_\theta = \frac{1}{2}\langle -1, 0, 1\rangle$. This points upward whereas outward on this cone points down. Hence we have the opposite orientation to the one we want.

$$\iint\limits_{T_1} \mathbf{F}\bullet d\mathbf{S} = -\iint\limits_{D_1} -\left(\frac{\sqrt{2}}{4}\right)\rho^2\,dA = \iint\limits_{D_1}\left(\frac{\sqrt{2}}{4}\right)\rho^2\,dA$$

Step 4_1: Do the double integral.

$$\iint\limits_{T_1} \mathbf{F} \bullet d\mathbf{S} = \left(\iint\limits_{\substack{0\leqslant\theta\leqslant 2\pi \\ 0\leqslant\rho\leqslant 2\sqrt{2}}} \frac{\sqrt{2}}{4} \right) \rho^2\, dA = \frac{\sqrt{2}}{4} \int_0^{2\pi} \int_0^{2\sqrt{2}} \rho^2 d\rho\, d\theta = \frac{\sqrt{2}}{4} \int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^{2\sqrt{2}}\, d\theta = \int_0^{2\pi} \frac{8}{3}\, d\theta = \frac{16\pi}{3}$$

Repeat for T_2

Step 2_2: Compute the normal field for $\mathbf{r}_2$.

$$(\mathbf{r}_2)_x = \langle 1,0,0\rangle = \mathbf{i}$$
$$(\mathbf{r}_2)_y = \langle 0,1,0\rangle = \mathbf{j}$$
$$(\mathbf{r}_2)_x \times (\mathbf{r}_2)_y = \mathbf{k} = \langle 0,0,1\rangle$$

Step 3_2: Reduce to a double integral.

$$\mathbf{F} \bullet \big((\mathbf{r}_2)_x \times (\mathbf{r}_2)_y\big) = \langle x,y,0\rangle \bullet \langle 0,0,1\rangle = 0$$

$$\iint\limits_{T_2} \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{D_2} 0\, dA = 0$$

and there was no need to check the orientation. Note however that it points up and up is the correct direction. Also there is no need for a separate step to evaluate the double integral.

Final Step: $\displaystyle\iint\limits_{T} \mathbf{F} \bullet d\mathbf{S} = \frac{16\,\pi}{3}$.

Example: Let T the unit cube in the first octant. Orient the surface so that the normal is outward from the solid cube. Integrate the field $\mathbf{F} = \langle x,y,z\rangle$ over this surface.

Step 1: There are six surfaces to parametrize. Let D be the unit square in the first quadrant of uw space. The six faces of the cube can be parametrized by

(1) C_1: $\mathbf{r}^{[0]}(u,w) = \langle u,w,0\rangle$;
(2) C_2: $\mathbf{r}^{[1]}(u,w) = \langle u,w,1\rangle$;
(3) C_3: $\mathbf{s}^{[0]}(u,w) = \langle u,0,w\rangle$;
(4) C_4: $\mathbf{s}^{[1]}(u,w) = \langle u,1,w\rangle$;
(5) C_5: $\mathbf{t}^{[0]}(u,w) = \langle 0,u,w\rangle$;
(6) C_6: $\mathbf{t}^{[1]}(u,w) = \langle 1,u,w\rangle$

for $(u,w) \in D$.

Step 2:

(1) $\mathbf{r}_u^{[0]}(u,w) = \langle 1,0,0\rangle$; $\mathbf{r}_w^{[0]}(u,w) = \langle 0,1,0\rangle$; $\mathbf{r}_u^{[0]} \times \mathbf{r}_w^{[0]} = \langle 0,0,1\rangle$
(2) $\mathbf{r}_u^{[1]}(u,w) = \langle 1,0,0\rangle$; $\mathbf{r}_w^{[1]}(u,w) = \langle 0,1,0\rangle$; $\mathbf{r}_u^{[1]} \times \mathbf{r}_w^{[1]} = \langle 0,0,1\rangle$
(3) $\mathbf{s}_u^{[0]}(u,w) = \langle 1,0,0\rangle$; $\mathbf{s}_w^{[0]}(u,w) = \langle 0,0,1\rangle$; $\mathbf{s}_u^{[0]} \times \mathbf{s}_w^{[0]} = \langle 0,-1,0\rangle$
(4) $\mathbf{s}_u^{[1]}(u,w) = \langle 1,0,0\rangle$; $\mathbf{s}_w^{[1]}(u,w) = \langle 0,0,1\rangle$; $\mathbf{s}_u^{[1]} \times \mathbf{s}_w^{[1]} = \langle 0,-1,0\rangle$
(5) $\mathbf{t}_u^{[0]}(u,w) = \langle 0,1,0\rangle$; $\mathbf{t}_w^{[0]}(u,w) = \langle 0,0,1\rangle$; $\mathbf{t}_u^{[0]} \times \mathbf{t}_w^{[0]} = \langle 1,0,0\rangle$
(6) $\mathbf{t}_u^{[1]}(u,w) = \langle 0,1,0\rangle$; $\mathbf{t}_w^{[1]}(u,w) = \langle 0,0,1\rangle$; $\mathbf{t}_u^{[1]} \times \mathbf{t}_w^{[1]} = \langle 1,0,0\rangle$

Steps 3 & 4: Reduce to a double integral.

(1) $\mathbf{F} \bullet \big(\mathbf{r}_u^{[0]} \times (\mathbf{r}_w^{[0]}\big) = \langle u,w,0\rangle \bullet \langle 0,0,1\rangle = 0$; $\displaystyle\iint\limits_{C_1} \mathbf{F} \bullet d\mathbf{S} = \iint\limits_{D} 0\, dA = 0$

(2) $\mathbf{F} \bullet \left(\mathbf{r}_u^{[1]} \times (\mathbf{r}_w^{[1]}\right) = \langle u, w, 1\rangle \bullet \langle 0,0,1\rangle = 1; \displaystyle\iint_{C_2} \mathbf{F} \bullet d\mathbf{S} = \iint_D 1\, dA = \text{Area of}(D) = 1$

(3) $\mathbf{F} \bullet \left(\mathbf{s}_u^{[0]} \times (\mathbf{s}_w^{[0]}\right) = \langle u, 0, w\rangle \bullet \langle 0,-1,0\rangle = 0; \displaystyle\iint_{C_3} \mathbf{F} \bullet d\mathbf{S} = \iint_D 0\, dA = 0$

(4) $\mathbf{F} \bullet \left(\mathbf{s}_u^{[1]} \times (\mathbf{s}_w^{[1]}\right) = \langle u, 1, w\rangle \bullet \langle 0,-1,0\rangle = -1; \displaystyle\iint_{C_4} \mathbf{F} \bullet d\mathbf{S} = \iint_D (-1)\, dA = -1$

(5) $\mathbf{F} \bullet \left(\mathbf{t}_u^{[0]} \times (\mathbf{t}_w^{[0]}\right) = \langle 0, u, w\rangle \bullet \langle 1,0,0\rangle = 0; \displaystyle\iint_{C_5} \mathbf{F} \bullet d\mathbf{S} = \iint_D 0\, dA = 0$

(6) $\mathbf{F} \bullet \left(\mathbf{t}_u^{[1]} \times (\mathbf{t}_w^{[1]}\right) = \langle 1, u, w\rangle \bullet \langle 1,0,0\rangle = 1; \displaystyle\iint_{C_6} \mathbf{F} \bullet d\mathbf{S} = \iint_D 1\, dA = 1$

Step 4^+**:** Determine the normal vectors and add up the answers.

(1) The normal vector points up, we want down.
(2) The normal vector points up, we want up.
(3) The normal vector points in negative y direction, we want negative.
(4) The normal vector points in negative y direction, we want positive.
(5) The normal vector points in positive x direction, we want negative.
(6) The normal vector points in positive x direction, we want positive.

Hence

$$\iint_T \mathbf{F} \bullet d\mathbf{S} = (-0) + 1 + 0 - (-1) - (0) + 1 = 3$$