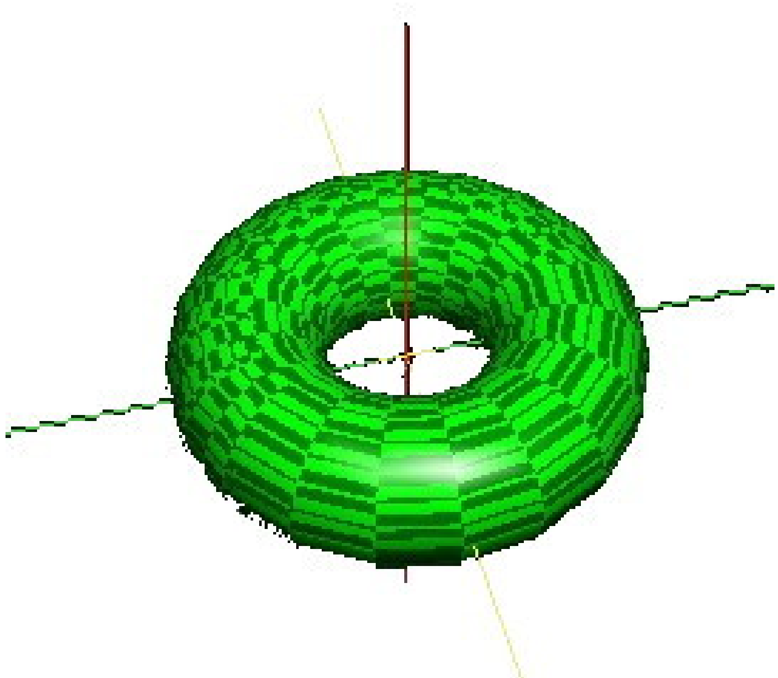
1. Example of the Divergence Theorem

Example. A torus

$$\mathbf{r}(u,w) = \Big\langle \big(a + b\cos(w)\big)\cos(u), \big(a + b\cos(w)\big)\sin(u), b\sin(w)\Big\rangle\,; \qquad 0 \leqslant u \leqslant 2\pi, \quad 0 \leqslant w \leqslant 2\pi$$

with $a > b > 0$.



Torus is $a = 2$, $b = 1$

Note

$$\mathbf{r}_u(u,w) = \big\langle -\big(a + b\cos(w)\big)\sin(u), \big(a + b\cos(w)\big)\cos(u), 0\big\rangle$$

$$\mathbf{r}_w(u,w) = \langle -b\sin(w)\cos(u), -b\sin(w)\sin(u), b\cos(w)\rangle$$

$$\mathbf{r}_u \times \mathbf{r}_w = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\big(a + b\cos(w)\big)\sin(u) & \big(a + b\cos(w)\big)\cos(u) & 0 \\ -b\sin(w)\cos(u) & -b\sin(w)\sin(u) & b\cos(w) \end{vmatrix} =$$

$$\Big\langle b\cos(w)\cos(u)\big(a + b\cos(w)\big), b\cos(w)\sin(u)\big(a + b\cos(w)\big), b\sin(w)\big(a + b\cos(w)\big)\Big\rangle =$$

$$\mathbf{r}_u \times \mathbf{r}_w = b\big(a + b\cos(w)\big)\,\langle \cos(w)\cos(u), \cos(w)\sin(u), \sin(w)\rangle$$

$$|\mathbf{r}_u \times \mathbf{r}_w| = b\big(a + b\cos(w)\big)$$

The torus is the boundary of a solid and so is orientable. At $(0,0)$ the point is $\mathbf{r}(0,0) = \langle a + b, 0, 0\rangle$ which is the largest of the four points on the x-axis. (The other three are $\langle a - b, 0, 0\rangle$, $\langle -a + b, 0, 0\rangle$ and $\langle -a - b, 0, 0\rangle$.)

At $(0,0)$, $\mathbf{r}_u \times \mathbf{r}_w(0,0) = b(a + b)\,\langle 1, 0, 0\rangle$ and since $b(a + b) > 0$, this normal field points out.

There is no need to check additional points but suppose you were asked to check the direction at $\langle a - b, 0, 0\rangle$. Now the first problem is to find (u,w) so that $\mathbf{r}(u,w) = \langle a - b, 0, 0\rangle$. To get the z-coordinate right, $w = 0$ or π or 2π. $w = 0$ or 2π gives an $a + b$ factor in the other two coordinates so $w = \pi$ looks like the right way to go. To get the y-coordinate right, take $u = 0$ and we see

$\mathbf{r}(0,\pi) = \langle a-b, 0, 0\rangle$. At this point, $\mathbf{r}_u \times \mathbf{r}_w(0,\pi) = a(a-b)\langle -1, 0, 0\rangle$. Since $a(a-b) > 0$, this points towards the origin, which is outward from the solid!

Surface area of a torus: $|\mathbf{r}_u \times \mathbf{r}_w| = b\big(a + b\cos(w)\big)$ with $0 \leqslant u \leqslant 2\pi$, $0 \leqslant w \leqslant 2\pi$.
Hence

$$\text{Surface area} = \iint\limits_{\substack{0\leqslant u\leqslant 2\pi\\ 0\leqslant w\leqslant 2\pi}} b\big(a+b\cos(w)\big)\,dA = \int\limits_0^{2\pi}\int\limits_0^{2\pi} b\big(a+b\cos(w)\big)\,dw\,du =$$

$$b\int\limits_0^{2\pi} aw + b\sin(w)\Big|_0^{2\pi}\,du = b\int\limits_0^{2\pi} 2\pi a\,du = 4\pi^2 ab$$

Volume of a torus: If E is the solid torus, the volume is $\iiint_E 1\,dV$. The fields $\mathbf{F}_1 = \langle x, 0, 0\rangle$, $\mathbf{F}_2 = \langle 0, y, 0\rangle$ and $\mathbf{F}_3 = \langle 0, 0, z\rangle$ have the property that $\operatorname{div}\mathbf{F}_i = 1$. Hence

$$\iint_{\partial E} \mathbf{F}_i \bullet d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}_i\,dV = \iiint_E 1\,dV$$

$$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_w)\,dA = b\big(a+b\cos(w)\big)\,\langle\cos(w)\cos(u), \cos(w)\sin(u), \sin(w)\rangle\,dA$$

Of the three fields, $\mathbf{F}_3$ seems easiest to work with.

$$\mathbf{F}_3 = \langle 0, 0, b\sin(w)\rangle$$

so

$$\mathbf{F}_3 \bullet d\mathbf{S} = \big(b\big(a+b\cos(w)\big)\sin(w)\cdot b\sin(w)\big)\,dA = b^2(a\sin^2(w) + b\cos(w)\sin^2(w)\,dA$$

Hence

$$\iiint_E 1\,dV = \iint\limits_{\substack{0\leqslant u\leqslant 2\pi\\ 0\leqslant w\leqslant 2\pi}} b^2\Big(a\sin^2(w) + b\cos(w)\sin^2(w)\Big)\,dA =$$

$$\int_0^{2\pi}\int_0^{2\pi} b^2\Big(a\sin^2(w) + b\cos(w)\sin^2(w)\Big)\,du\,dw = 2\pi b^2\int_0^{2\pi}(a\sin^2(w) + b\cos(w)\sin^2(w)\,dw =$$

$$2\pi b^2\int_0^{2\pi} a\sin^2(w)\,dw = 2\,\pi b^2 a\frac{1-\cos(2w)}{2}\Bigg|_0^{2\pi} = 2\pi^2 ab^2$$