1. Vectors perpendicular to other vectors.

Sometimes it is necessary to find some vector perpendicular to other vectors. There are many choices but typically any one will do.

Given two vectors in 3-space, say $\mathbf{v}_1$ and $\mathbf{v}_2$ then

$$\mathbf{v}_1 \times \mathbf{v}_2$$

is a vector perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. If the cross product is $\mathbf{0}$ then $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel.

Given a vector $\mathbf{v} = \langle a, b\rangle$ a non-zero perpendicular vector is $\langle -b, a\rangle$.

Given a vector $\mathbf{v} = \langle a, b, c\rangle$ suppose $a \neq 0$. Then a non-zero perpendicular vector $\mathbf{N} = \langle -b, a, 0\rangle$. If $b \neq 0$, $\mathbf{N} = \langle -b, a, 0\rangle$ is perpendicular and non-zero. If $c \neq 0$, $\mathbf{N} = \langle 0, -c, b\rangle$ is perpendicular and non-zero.

2. Lines in 2 or 3-space

A line is determined by point, $\mathbf{p}$, and vector, $\mathbf{v}$, parallel to the line. A *vector equation* for the line is

$$\mathbf{p} + t\mathbf{v}$$

The *parametric equation* just writes the vector equation in coordinates.

Given two points P_0, P_1, $\mathbf{p} = P_0$ and $\mathbf{v} = P_1 - P_0$ amongst other choices.

In general, a vector $\mathbf{w}$ lies in the line $\mathbf{p} + t\mathbf{v}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel. A point P_0 lies on the line $\mathbf{p} + t\mathbf{v}$ if and only if you can solve $\mathbf{p} + t\mathbf{v} = P_0$ if and only if $\mathbf{\Delta p} = \mathbf{p} - P_0$ is parallel to $\mathbf{v}$.

The parametric equation can also be interpreted as describing the motion of a particle moving along the line. These two interpretations give two problems:

$$\mathbf{p}_0 + t\mathbf{v}_0 = \mathbf{p}_1 + t\mathbf{v}_1$$

the *collision problem* and

$$\mathbf{p}_0 + t\mathbf{v}_0 = \mathbf{p}_1 + s\mathbf{v}_1$$

the *intersection problem.*

2.1. The normal equation for a line in the plane.

A normal vector to a line in the plane is a non-zero vector perpendicular to the line, or equivalently perpendicular to $\mathbf{v}$. If $\mathbf{N}$ is a normal vector, the line can be described as all points which satisfy

$$\mathbf{N} \bullet \langle x, y\rangle = d$$

and a point $\mathbf{P} = \langle x_0, y_0\rangle$ is on the line if and only if $\mathbf{N} \bullet \mathbf{P} = d$.

The old $y = mx + b$ becomes $\langle 0, b\rangle + t\langle 1, m\rangle$.

2.2. **The intersection problem reprised.**

In general, if you want to solve the intersection problem between a parametric line $\mathbf{p} + t\mathbf{v}$ and a normal equation for a line $\mathbf{N} \bullet \langle x, y \rangle = d$ plug the parametric equation into the normal equation:

$$\mathbf{N} \bullet (\mathbf{p} + t\,\mathbf{v}) = d$$

Using your rules for dot products, we get

$$\mathbf{N} \bullet \mathbf{p} + t\,(\mathbf{N} \bullet \mathbf{v}) = d \tag{$*$}$$

The procedure we used to get ($*$) all you really need to remember, the rest of the discussion in this section is to see why it works. If you do what ($*$) says you have an equation of the form $(number) + (number)t = number$ and you have been solving these for years.

Equation ($*$) has a unique solution if and only if $\mathbf{N} \bullet \mathbf{v} \neq 0$. If $\mathbf{N} \bullet \mathbf{v} = 0$ then $\mathbf{N}$ is a normal vector to the parametric line and so the two lines are parallel. If $\mathbf{N} \bullet \mathbf{p} = d$ the two lines are identical and if not they are parallel and never intersect. If $\mathbf{N} \bullet \mathbf{v} \neq 0$, the point of intersection is

$$\mathbf{I} = \mathbf{p} + \left(\frac{d - \mathbf{N} \bullet \mathbf{p}}{\mathbf{N} \bullet \mathbf{v}} \right) \mathbf{v}$$

2.3. **A line perpendicular to a given line through a given point.**

Let $\mathbf{p}$ be a point and a line $\mathbf{N} \bullet \langle x, y \rangle = d$ with $\mathbf{p}$ not on the line. This last condition is equivalent to $\mathbf{N} \bullet \mathbf{p} \neq d$. We will write down a parametric equation for the line. Since $\mathbf{p}$ is a point on the line, we have a point so we need a vector parallel to the line. But $\mathbf{N}$ is such a vector since the line we are looking for and our given line are perpendicular. Then the line is $\mathbf{p} + t\mathbf{N}$. This line intersects the original line at

$$\mathbf{I} = \mathbf{p} + \left(\frac{d - \mathbf{N} \bullet \mathbf{p}}{\mathbf{N} \bullet \mathbf{N}} \right) \mathbf{N}$$

The distance from the point to the line is the length of $\mathbf{p} - \mathbf{I}$ or

$$\frac{|\, d - \mathbf{N} \bullet \mathbf{p} \,|}{|\mathbf{N}|}$$

3. Planes in 3-space.

A plane is determined by a point $\mathbf{p}$ and two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with $\mathbf{v}_1$ not parallel to $\mathbf{v}_2$.

$$\mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2$$

This is a *parametric equation* of the plane.

Given 3 points Q, P_1 and P_2, $Q = \mathbf{p}$, $\mathbf{v}_1 = P_1 - Q$ and $\mathbf{v}_2 = P_2 - Q$ amongst other choices. If $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$, Q, P_1 and P_2 are *collinear*.

A plane is also determined by a normal vector $\mathbf{N} = \langle a, b, c \rangle$ and and a point (x_0, y_0, z_0)

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$. Can also write this as

$$\mathbf{N} \bullet \langle x, y, z \rangle = d$$

This last equation is called the *vector equation* for the plane.

Notice two planes are parallel if and only if any two normal vectors are parallel. A vector lies in a plane if and only if it is perpendicular to a normal vector to the plane. A point $\mathbf{p}_1$ lies in the plane $\mathbf{N} \bullet \mathbf{p}_1 = d$ if and only if $\boldsymbol{\Delta}\mathbf{p} = \mathbf{p}_1 - \mathbf{p}$ is perpendicular to a normal vector to the plane.

3.1. Parametric to vector equation for a plane.

Given $\mathbf{v}_1$ and $\mathbf{v}_2$ we can take $\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel if $\mathbf{N} \neq \mathbf{0}$. Of course we may take $\mathbf{p}$ for a point on the plane.

3.2. Is a point on a plane?

Given a point, (x_1, y_1, z_1) it is easy to check whether $ax_1 + by_1 + cz_1 = d$ or not. Equivalently, let $\mathbf{P}_1 = \langle x_1, y_1, z_1 \rangle$ and it is easy to check whether $\mathbf{N} \bullet \mathbf{P}_1 = d$ or not.

It is harder to find s and t such that $\mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2 = \mathbf{P}_1$. This is a set of three equations in two unknowns and you do know how to deal with such a problem. If you just need a yes or no answer check $\boldsymbol{\Delta}\mathbf{p}$ is perpendicular to $\mathbf{N}$.

3.3. Find a point in a plane.

This is easy if the plane is given parametrically: $\mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2$. Just take any two values for s and t you fancy. Zero-zero is a popular choice so $\mathbf{p}$ is in the plane.

If the plane is given as $\mathbf{N} \bullet \langle x, y, z \rangle = d$ let $\mathbf{N} = \langle a, b, c \rangle$. Then $\left(\dfrac{d}{a}, 0, 0\right)$ is in the plane if $a \neq 0$, $\left(0, \dfrac{d}{b}, 0\right)$ is in the plane if $b \neq 0$ and $\left(0, 0, \dfrac{d}{c}\right)$ is in the plane if $c \neq 0$. Since $\mathbf{N} \neq \mathbf{0}$ at least one of the above formulas gives a point.

More generally, if $a \neq 0$ then for any pair (y_0, z_0) there is a point (x_0, y_0, z_0) on the plane with $x_0 = \dfrac{d - b\,y_0 - c\,z_0}{a}$. Similar results hold if $b \neq 0$ or $c \neq 0$.

If you are writing a computer program or if for some other reason you want a formula for a point that does not require thought, note $\left(\dfrac{d}{\mathbf{N} \bullet \mathbf{N}}\right)\mathbf{N}$ is such a point.

3.4. Find an equation of a line perpendicular to a plane through a point.

Let the plane be given by $\mathbf{N} \bullet \langle x, y, z \rangle = d$ and let $\mathbf{p}$ be the point.

For the line, we need a point and a vector parallel to the line. For the point take, $\mathbf{p}$. Note that $\mathbf{N}$ is perpendicular to the plane and hence parallel to the line. Hence an equation for the line is

$$\mathbf{N}t + \mathbf{p}$$

3.5. **Do a line and a plane intersect and if so, where?**

Let $\mathbf{p} + t\mathbf{v}$ be the line and $\mathbf{N} \bullet \langle x, y, z\rangle = d$ the plane. A point $\mathbf{P}$ is in the plane if and only if $\mathbf{N} \bullet \mathbf{P} = d$ and if $\mathbf{P} = \mathbf{p} + t\mathbf{v}$ this shows $\mathbf{N} \bullet (\mathbf{p} + t\mathbf{v}) = d$ for any value of t for which $\mathbf{p} + t\mathbf{v}$ is in the plane. Now

$$\mathbf{N} \bullet \mathbf{p} + t\ (\mathbf{N} \bullet \mathbf{v}) = d$$

If $\mathbf{N} \bullet \mathbf{v} \neq 0$ there is a unique solution. If $\mathbf{N} \bullet \mathbf{v} = 0$ there is no solution if $d \neq \mathbf{N} \bullet \mathbf{p}$ or the line lies in the plane if $d = \mathbf{N} \bullet \mathbf{p}$.

Note the similarity between this problem and §1.2.

3.6. **Distance from a point to a plane.** If the plane is given by $\mathbf{N} \bullet \langle x, y, z\rangle = d$ and $\mathbf{p}$ is a point, the distance from the point to the plane is

$$\frac{|\mathbf{N} \bullet \mathbf{p} - d|}{|\mathbf{N}|}$$

To see this, note the line perpendicular to the plane through $\mathbf{p}$ found above intersects the plane when $\mathbf{N} \bullet (\mathbf{N}t + \mathbf{p}) = d$ or when $t = \frac{d - \mathbf{N}\bullet\mathbf{p}}{\mathbf{N}\bullet\mathbf{N}}$. The point of intersection is $\frac{d - \mathbf{N}\bullet\mathbf{p}}{\mathbf{N}\bullet\mathbf{N}}\mathbf{N} + \mathbf{p}$ and the distance from this point to $\mathbf{p}$ is

$$\left|\left(\frac{d - \mathbf{N} \bullet \mathbf{p}}{\mathbf{N} \bullet \mathbf{N}}\mathbf{N} + \mathbf{p}\right) - \mathbf{p}\right|$$

3.7. **Do two planes intersect?**

Let the two planes be $\mathbf{N}_1 \bullet \langle x, y, z\rangle = d_1$ and $\mathbf{N}_2 \bullet \langle x, y, z\rangle = d_2$. The intersection may be empty, so the planes are parallel. It may be a plane, in which case the planes are identical. Finally, the intersection is usually a line so the problem is to determine the line, always allowing for the fact that you may be in one of the degenerate situations.

Step 1: Vector parallel to line, $\mathbf{v}$. Well $\mathbf{v}$ lies in the first plane so $\mathbf{N}_1 \bullet \mathbf{v} = 0$ and $\mathbf{v}$ lies in the second plane so $\mathbf{N}_2 \bullet \mathbf{v} = 0$ so we may take

$$\mathbf{v} = \mathbf{N}_1 \times \mathbf{N}_2$$

If $\mathbf{v} = \mathbf{0}$, the two normals are parallel and so are the two planes and we are in one of the two degenerate situations. Assuming neither normal is $\mathbf{0}$, $\mathbf{N}_1 = c\mathbf{N_2}$ and so the planes coincide if and only if $d_1 = cd_2$ for the same c.

Step 2. Assuming $\mathbf{v} = \langle \alpha, \beta, \gamma\rangle \neq \mathbf{0}$, we still need to find a point which lies on both planes. A good way to do this is to set one of the variables equal to 0 and solve for the two equations $\mathbf{N}_1 \bullet \langle x, y, z\rangle = d_1$ $\mathbf{N}_2 \bullet \langle x, y, z\rangle = d_2$ for the remaining two variables. If you choose the variable to be set to 0 as in the bullet points, you are guaranteed a solution.

- If $\alpha \neq 0$, set $x = 0$.
- If $\beta \neq 0$, set $y = 0$.
- If $\gamma \neq 0$, set $z = 0$.

3.8. **If the planes do intersect, what is the angle of intersection?**

From solid geometry, it is the same as the angle between the normals so

$$\cos\theta = \frac{\mathbf{N}_1 \bullet \mathbf{N}_2}{|\mathbf{N}_1| \cdot |\mathbf{N}_2|}$$

3.9. **What is the distance between two parallel planes?** Suppose the two equations are $\mathbf{N}_1 \bullet \langle x, y, z\rangle = d_1$ and $\mathbf{N}_2 \bullet \langle x, y, z\rangle = d_2$. The two planes are parallel if and only if there is a number γ such that $\mathbf{N}_1 = \mathbf{N}_2\gamma$. Then another equation for the second plane is $\gamma\mathbf{N}_2 \bullet \langle x, y, z\rangle = \gamma d_2$ or $\mathbf{N}_1 \bullet \langle x, y, z\rangle = \gamma d_2$. Pick a point in the second plane and compute the distance to the first. The answer does not involve coordinates for the point.

$$\frac{|d_1 - \gamma d_2|}{|\mathbf{N}_1|}$$

3.10. **Given two lines is there a plane containing them?**

Let $\mathbf{p}_1 + t\mathbf{v}_1$ and $\mathbf{p}_2 + t\mathbf{v}_2$ be the two lines. If there is such a plane, then its normal vector is perpendicular to each $\mathbf{v}_i$ so try $\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2$. If this is 0, then the lines are parallel but they may still determine a plane. If they do then $\mathbf{p}_2 - \mathbf{p}_1$ also lies in the plane so try $\mathbf{N} = \mathbf{v}_1 \times (\mathbf{p}_2 - \mathbf{p}_1)$. If this is also 0 then the two lines are identical and they do not determine a plane.

If one of these two calculations yields a non-zero normal $\mathbf{N}$ then pick any point on either line say $\mathbf{p}_1$ so we have a plane

$$\mathbf{N} \bullet \langle x, y, z\rangle = \mathbf{N} \bullet \mathbf{p}_1$$

It may still happen that the second line fails to lie in this plane. Since $\mathbf{N} \bullet \mathbf{v}_2 = 0$, the second line either lies in this plane or misses it entirely so there is a plane containing the two lines if and only if

$$\mathbf{N} \bullet \mathbf{p}_1 = \mathbf{N} \bullet \mathbf{p}_2$$

for some non-zero normal vector $\mathbf{N}$.

3.11. **A quick test for skew lines.**

Two lines $\mathbf{p}_1 + t\mathbf{v}_1$ and $\mathbf{p}_2 + t\mathbf{v}_2$ are *skew* provided they are not parallel and they do not intersect. If they are not parallel, $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ so $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is a necessary condition for two lines to be skew. If both lines lie in a plane we know two lines in a plane intersect unless they are parallel, so if $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ the lines are skew if and only if

$$\mathbf{p}_1 \bullet (\mathbf{v}_1 \times \mathbf{v}_2) \neq \mathbf{p}_2 \bullet (\mathbf{v}_1 \times \mathbf{v}_2)$$

Both sides of this equation are triple products and can be evaluated via a determinant. It is usually faster to compute the triple product below. Summarizing, two lines are skew if and only if

$$(\mathbf{p}_1 - \mathbf{p}_2) \bullet (\mathbf{v}_1 \times \mathbf{v}_2) \neq 0$$

3.12. **A plane containing a line and a point not on the line.**

Let $\mathbf{p}_1 + t\mathbf{v}_1$ be the line and $\mathbf{p}_2$ be the point not on the line. We need to find a normal vector $\mathbf{N}$. Since the line lies in the plane, $\mathbf{v}_1 \bullet \mathbf{N} = 0$. Since both $\mathbf{p}_1$ and $\mathbf{p}_2$ are points in the plane, $\mathbf{p}_1 - \mathbf{p}_2$ is a vector in the plane and so a choice for $\mathbf{N}$ is $\mathbf{v}_1 \times (\mathbf{p}_1 - \mathbf{p}_2)$ if it is non-zero. But if it is zero, $\mathbf{p}_1 - \mathbf{p}_2$ is a multiple of $\mathbf{v}_1$ and then $\mathbf{p}_2$ lies on the line. Hence an equation for the plane is

$$\big(\mathbf{v}_1 \times (\mathbf{p}_1 - \mathbf{p}_2)\big) \bullet \langle x, y, z\rangle = \big(\mathbf{v}_1 \times (\mathbf{p}_1 - \mathbf{p}_2)\big) \bullet \mathbf{p}_2$$

If you want to compute $\big(\mathbf{v}_1 \times (\mathbf{p}_1 - \mathbf{p}_2)\big) \bullet \mathbf{p}_2$ use properties of the dot and cross product to see

$$\big(\mathbf{v}_1 \times (\mathbf{p}_1 - \mathbf{p}_2)\big) \bullet \mathbf{p}_2 = (\mathbf{v}_1 \times \mathbf{p}_1) \bullet \mathbf{p}_2$$

This is also the same as $-(\mathbf{v}_1 \times \mathbf{p}_2) \bullet \mathbf{p}_1$ which is what you get naturally on the right hand side of you use $\mathbf{p}_1$ as your point.

3.13. **Plane perpendicular to a line through a point.**

Let $\mathbf{p}_0 + t\mathbf{v}$ be the line and $\mathbf{p}_1$ be the point. Then $\mathbf{v}$ is perpendicular to the plane so we can take $\mathbf{v}$ for the normal vector to the plane. Hence the equation is

$$\mathbf{v} \bullet \langle x, y, z\rangle = \mathbf{v} \bullet \mathbf{p}_1$$

3.14. **Going from a vector equation to a parametric equation.**

Suppose we are given a plane $\mathbf{N} \bullet \langle x, y, z\rangle = d$ and we need to find a parametric equation for it. We need two vectors parallel to the plane and a point in the plane. From the above discussion take $\mathbf{P}_0 = \dfrac{d}{\mathbf{N} \bullet \mathbf{N}}\,\mathbf{N}$ to be the point.

Use §1 to find one vector perpendicular to $\mathbf{N}$, say $\mathbf{v}_1$. Then use $\mathbf{v}_2 = \mathbf{N} \times \mathbf{v}_1$, so

$$\mathbf{P}_0 + t\mathbf{v}_1 + s\mathbf{v}_2$$

is the parametric equation for the plane.

3.15. **Sided-ness of planes.**

The plane $\mathbf{N} \bullet \langle x, y, z\rangle = d$ divides all of three space into three parts. As an example, the xy-plane divides three space into the xy-plane, the points *above* the xy plane and the points *below* the xy plane.

In general, above and below have no meaning, but we may proceed as follows. Let $\mathbf{P} = \langle x_0, y_0, z_0\rangle$ be a point in three space. Then $\mathbf{N} \bullet \mathbf{P} - d$ is number equal to 0, greater than 0 or less than 0. If it is equal to 0, $\mathbf{P}$ lies in the plane. If it is greater than 0 then $\mathbf{P}$ lies in that half of three space pointed *into* by the normal vector. If it is less than 0 then $\mathbf{P}$ lies in that half of three space pointed *out of* by the normal vector.

If we take $\mathbf{N} = \mathbf{k}$ for the xy-plane, then $\mathbf{N}$ points into the upper portion of three space and $\mathbf{N}$ points out of the lower portion. If we take $\mathbf{N} = -\mathbf{k}$ then $\mathbf{N}$ points out of the upper half space.