1. The one constraint theory

The problem is to find the extrema of a function $f(x, y)$ subject to the constraint $g(x, y) = c$. One place where this comes up is in solving max/min problems on bounded domains where the boundary curve is given as a level curve of a function $g(x, y)$.

The book gives a derivation of the theory below, but here is a second way to understand the result.

Parametrize the constraint curve by arc length $\mathbf{r}(s)$ so $g\big(\mathbf{r}(s)\big) = c$. Then we want to study the extrema of the function $h(s) = f\big(\mathbf{r}(s)\big)$. From first year calculus, we start by locating the critical points of h:

$$h'(s) = \nabla f\big(\mathbf{r}(s)\big) \bullet \mathbf{r}'(s) = 0$$

The immediate problem is that we don't know $\mathbf{r}'(s)$. However, from the constraint equation we know,

$$0 = g'(s) = \nabla g\big(\mathbf{r}(s)\big) \bullet \mathbf{r}'(s)$$

and since we are in the plane, $\mathbf{r}'(s)$ is a multiple of $\left\langle -\dfrac{\partial g}{\partial y}, \dfrac{\partial g}{\partial x} \right\rangle$, provided $\nabla g \neq \langle 0, 0\rangle$.

Hence the critical points of h occur when $\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y} = 0$ or

$$\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \tag{1}$$

Lagrange would tackle this problem by arguing via geometry that since ∇f and ∇g are both perpendicular to $\mathbf{r}'(s)$ and we are in the plane, ∇f and ∇g must be parallel, so

$$\nabla f = \lambda \nabla g \tag{2}$$

The number λ is the *Lagrange multiplier*. In most cases it is introduced to make the algebra easier but in certain situations it does have meaning for the problem. We will not pursue this here but see A, B, C or D if you want further information.

Equation (1) is symmetric in f and g whereas (2) is not. The derivation in the book makes it clear why (2) is the preferred equation rather than

$$\nabla g = \lambda \nabla f \tag{3}$$

although problem 23 in §14.8 gives an example where the minimum value occurs at a solution to (3). The solutions to equation (1) give all solutions to (2) and (3) but is harder to work with. Of course as long as there are no critical points of g on the constraint curve $g(x, y) = c$, (3) implies (2) so (1) and (2) are equivalent.

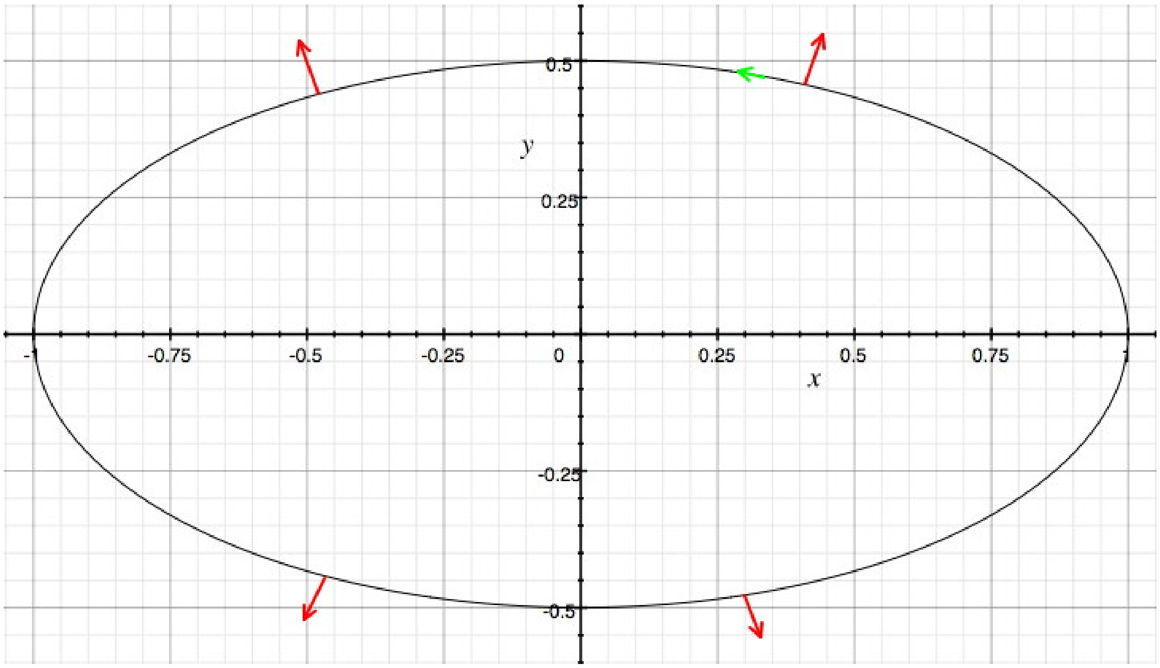
Here is a bit more of the geometry of the discussion above. If $\nabla g \neq \langle 0, 0\rangle$, then $\left\langle -\dfrac{\partial g}{\partial y}, \dfrac{\partial g}{\partial x} \right\rangle$ is parallel to $\mathbf{r}'(s)$. Using the right-hand rule, if you put your fingers along ∇g with your thumb up, your fingers curl towards $\left\langle -\dfrac{\partial g}{\partial y}, \dfrac{\partial g}{\partial x} \right\rangle$. Orient the curve at the point so that you are moving in the same direction as $\left\langle -\dfrac{\partial g}{\partial y}, \dfrac{\partial g}{\partial x} \right\rangle$. Declare this to be the *preferred direction for the curve at the* *point.* Then $h'(s)$ has the same sign as

$$H(x,y) = \det \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \tag{4}$$

So if we move in the preferred direction along the constraint curve $g(x,y) = c$, the value of f along the curve is

- *increasing* if $H(x,y) > 0$,
- *decreasing* if $H(x,y) < 0$ and
- *is a critical point* if $H(x,y) = 0$.

Example: Discuss the function $f(x, y) = e^{-xy}$ restricted to the curve $g(x, y) = x^2 + 4y^2 = 1$. Here is a picture of the constraint curve.



The red vectors point in the direction of the gradient ∇g and the green arrow indicates the preferred direction along the constraint curve.

- $\nabla g = \langle 2x, 8y\rangle$
- $\nabla f = \langle -ye^{-xy}, -xe^{-xy}\rangle$
- $H(x, y) = \det \begin{vmatrix} 2x & 8y \\ -ye^{-xy} & -xe^{-xy} \end{vmatrix}$

So for example, at $\left(\frac{3}{5}, \frac{2}{5}\right)$, $H\left(\frac{3}{5}, \frac{2}{5}\right) = \det \begin{vmatrix} \frac{6}{5} & \frac{16}{5} \\ -\frac{2}{5}e^{-6/25} & -\frac{3}{5}e^{-6/25} \end{vmatrix} = \left(\frac{-18}{25} + \frac{32}{25}\right) e^{-6/25} > 0$ so at the point $\left(\frac{3}{5}, \frac{2}{5}\right)$ on the constraint curve, the value of e^{-xy} is increasing as we go along the curve counterclockwise. At $(1, 0)$, $H(1, 0) = \det \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} < 0$ so at the point $(1, 0)$ on the constraint curve, the value of e^{-xy} is decreasing as we go along the curve counterclockwise.

Let us locate the critical points of the restricted problem using Lagrange multipliers. The three equations are

(1) $2x = \lambda(-ye^{-xy})$
(2) $8y = \lambda(-xe^{-xy})$
(3) $x^2 + 4y^2 = 1$

If $\lambda = 0$, $x = y = 0$ and we are not on the constraint curve. Hence we may assume hereafter that $\lambda \neq 0$.

If $x = 0$ (1) implies $y = 0$ since $\lambda \neq 0$. Hence $x \neq 0$. Similarly, if $y = 0$, (2) implies $x = 0$ since $\lambda \neq 0$. Hence $y \neq 0$.

Note (1) implies $2x^2 = \lambda(-xye^{-xy})$ and (2) implies $8y^2 = \lambda(-xye^{-xy})$ so $2x^2 = 8y^2$ or $x^2 = 4y^2$. Since $x^2 + 4y^2 = 1$, $2x^2 = 1$ and $x = \pm\frac{1}{\sqrt{2}}$. There are only 4 possible points

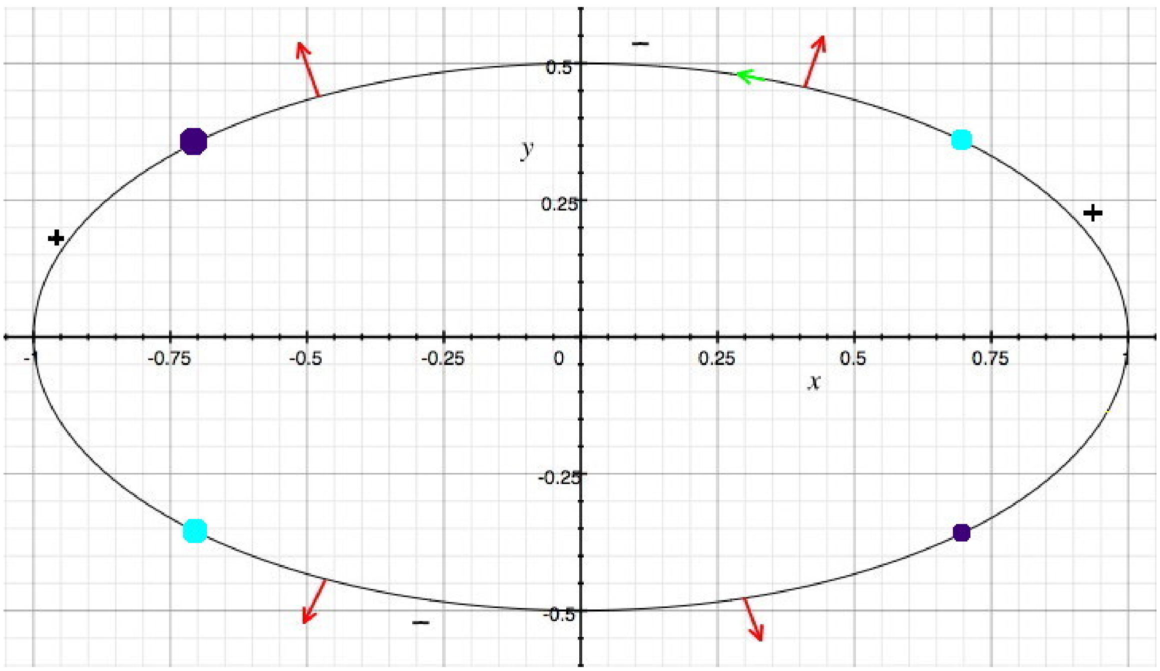
$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$$

Then

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{\frac{1}{4}}$ which is the maximum value

and

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{-\frac{1}{4}}$ which is the minimum value.



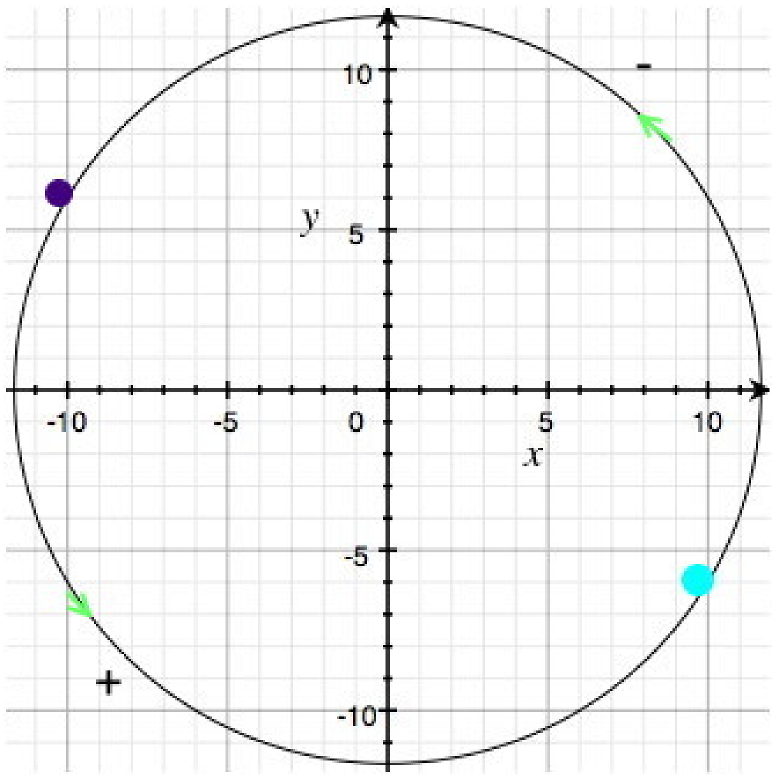
The light blue dots are the approximate locations of the maxima and the purple dots are the approximate locations of the minima. You can pick out the intervals where f is increasing and where it is decreasing as you move along the curve in the preferred direction and note that it is consistent with the calculations of H above.

We have put a minus sign next to the intervals on which f is decreasing and a plus sign next to the intervals on which f is increasing. This resembles the first year calculus technique where you draw a number line, mark the critical points and put a plus or minus in the appropriate intervals between the critical points.

Example: Discuss the function $f(x,y) = 5x - 3y$ restricted to the curve $g(x,y) = x^2 + y^2 = 136$.

- $\nabla g = \langle 2x, 2y \rangle$
- $\nabla f = \langle 5, -3 \rangle$
- $x^2 + y^2 = 136$
- $H(x,y) = \det \begin{vmatrix} 2x & 2y \\ 5 & -3 \end{vmatrix}$

Lagrange multipliers: $5 = \lambda 2x$, $-3 = \lambda 2y$ and $x^2 + y^2 = 136$. Hence $x = \dfrac{5}{2\lambda}$, $y = \dfrac{-3}{2\lambda}$ so $\left(\dfrac{5}{2\lambda}\right)^2 + \left(\dfrac{-3}{2\lambda}\right)^2 = \dfrac{25}{4\lambda^2} + \dfrac{9}{4\lambda^2} = \dfrac{34}{4\lambda^2} = 136$ so $\lambda^2 = \dfrac{34}{4 \cdot 136} = \dfrac{1}{16}$ so $\lambda = \pm\dfrac{1}{4}$. Hence there are two points $(10,-6)$ and $(-10,6)$. Hence $f(10,-6) = 68$ and $f(-10,6) = -68$.



Example: Discuss the function $f(x,y) = x^2 + y^2$ restricted to the curve $g(x,y) = x^2 + xy = 1$.

- $\nabla g = \langle 2x + y, x \rangle$
- $\nabla f = \langle 2x, 2y \rangle$
- $x^2 + xy = 1$
- $H(x,y) = \det \begin{vmatrix} 2x+y & x \\ 2x & 2y \end{vmatrix}$

Lagrange multipliers: $2x = \lambda(2x+y)$, $2y = \lambda x$ and $x^2 + xy = 1$.

If $x = 0$ there is no value of y so that $(0,y)$ is on the constraint curve. Hence $x \neq 0$ and then $\lambda \neq 0$ either.

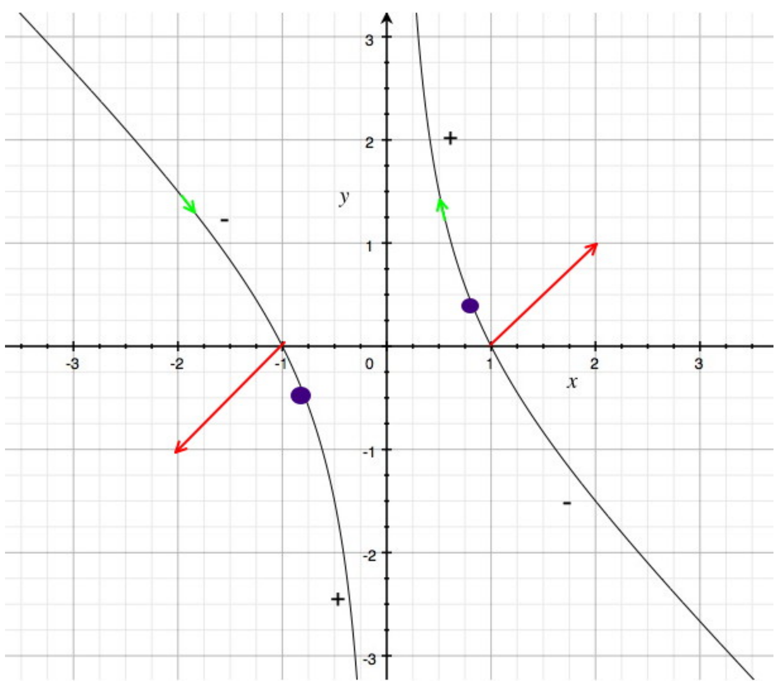
Now $2y = \lambda x$ so $2x = \lambda(2x + \lambda x/2)$ or $4x = x(4\lambda + \lambda^2)$. Since $x \neq 0$, $\lambda^2 + 4\lambda - 4 = 0$ or $\lambda = \dfrac{-4 \pm \sqrt{16+16}}{2} = -2 \pm 2\sqrt{2}$.

Then $y = (-1 \pm \sqrt{2}\,)x$ and $x^2 + (-1 \pm \sqrt{2}\,)x^2 = 1$. Hence $x^2(\pm\sqrt{2}\,) = 1$. There are only solutions for the plus sign and $x = \pm\dfrac{1}{\sqrt[4]{2}}$ and $y = (-1+\sqrt{2}\,)x$.

The critical points are

$$\left(\frac{1}{\sqrt[4]{2}}, \frac{-1+\sqrt{2}}{\sqrt[4]{2}}\right), \left(-\frac{1}{\sqrt[4]{2}}, \frac{1-\sqrt{2}}{\sqrt[4]{2}}\right)$$

and they are both minima.



Example: Find the maxima and minima of $f(x)$ restricted to the curve $y = 0$.

- $\nabla g = \langle 0, 1\rangle$
- $\nabla f = \langle f'(x), 0\rangle$

so $f'(x) = \lambda \cdot 0$ and $0 = \lambda \cdot 1$. Hence $f'(x) = 0$ and this really is max/min theory from first year calculus. Hence anything that can happen in the one-variable case can happen for Lagrange multipliers.

Example: Endpoints can matter with Lagrange multipliers. Find the extrema of $f(x, y) = x^2 + y^2$ restricted to the boundary of the lens-shaped region below $y = 2 - \dfrac{x^2}{8}$ and above $y = \dfrac{x^2}{8} - 2$.

The boundary is made up of two graphs

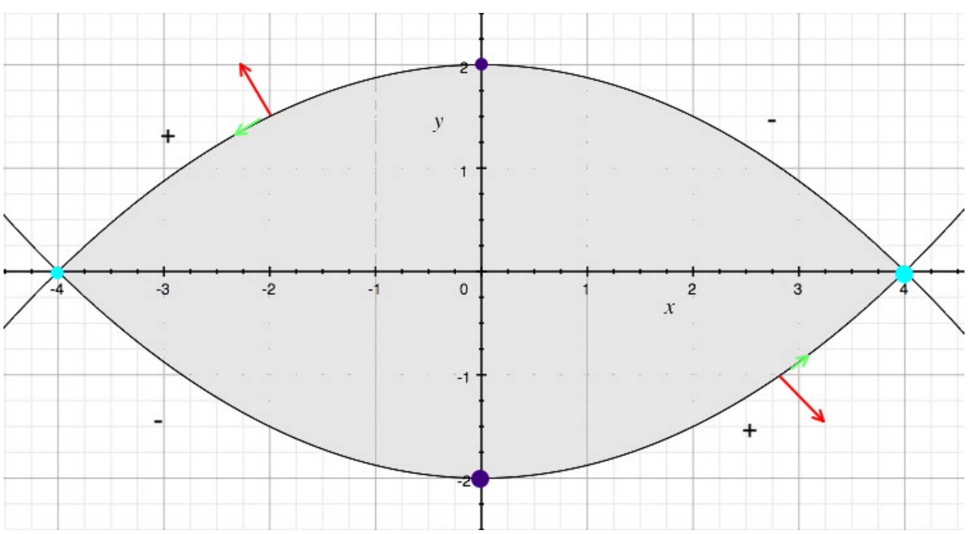
(1) the upper piece $y = 2 - \dfrac{x^2}{8}$, $-4 \leqslant x \leqslant 4$ or $g_1(x, y) = \dfrac{x^2}{8} + y = 2$
and
(2) the lower piece $y = \dfrac{x^2}{8} - 2$, $-4 \leqslant x \leqslant 4$ or $g_2(x, y) = \dfrac{x^2}{8} - y = 2$.

Start with the upper piece.

- $\nabla g_1 = \left\langle \dfrac{x}{4}, 1 \right\rangle$
- $\nabla f = \langle 2x, 2y\rangle$

$2x = \lambda \dfrac{x}{4}$, $2y = \lambda \cdot 1$ and $\dfrac{x^2}{8} + y = 2$. If $x = 0$, $y = 2$ and $\lambda = 4$ so $(0, 2)$ is a critical point. If $x \neq 0$, $\lambda = 8$ and $y = 4$. There are no x such that $(x, 4)$ lies on the parabola. Since $-4 \leqslant x \leqslant 4$, the two points $(\pm 4, 0)$ are also on the list.

The calculation for the lower piece is similar: $(0, -2)$ and $(\pm 4, 0)$ are the requisite points.



Since the boundary curve is close and bounded, you know there is a maximum value and it occurs at an absolute maxima. There is also a minimum value which occurs at an absolute minima.

The endpoints $x = \pm 4$ are the absolute maxima. If you don't do endpoints you may end up thinking that one of $y = \pm 2$ is the absolute maximum and the other is the absolute minimum.