1. The one constraint theory

The problem is to find the extrema of a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$.

The book gives a derivation of the theory below, but here is a second way to understand the result.

Parametrize the constraint surface $\mathbf{r}(s, t)$ so $g\big(\mathbf{r}(s, t)\big) = c$. Then we want to study the extrema of the function $h(s, t) = f\big(\mathbf{r}(s, t)\big)$. As we saw, this involves locating the critical points of h, the points where $\nabla h = \mathbf{0}$.

$$\nabla h(s,t) = \left\langle \nabla f\big(\mathbf{r}(s,t)\big) \bullet \frac{\partial\, \mathbf{r}(s,t)}{\partial\, s}, \nabla f\big(\mathbf{r}(s,t)\big) \bullet \frac{\partial\, \mathbf{r}(s,t)}{\partial\, t} \right\rangle = 0$$

The immediate problem is that we don't know $\dfrac{\partial\, \mathbf{r}(s,t)}{\partial\, s}$ or $\dfrac{\partial\, \mathbf{r}(s,t)}{\partial\, t}$, but we know they are perpendicular to the gradient of g. The critical point equation says ∇f is also perpendicular to $\dfrac{\partial\, \mathbf{r}(s,t)}{\partial\, s}$ and $\dfrac{\partial\, \mathbf{r}(s,t)}{\partial\, t}$ and since we are in three dimensions, the critical points of h occur when

$$\nabla f = \lambda \nabla g \tag{1}$$

This is not quite right since it is possible that ∇g might vanish at point on the constraint surface and these could be critical points of h.

> *To be sure you have all the critical points, find all solutions to $\nabla f = \lambda \nabla g$ and all solutions to $\nabla g = \boldsymbol{0}$ that lie on the critical surface.*

This works in any number of variables, not just 2 or 3. In this course we will not pursue how to determine what sort of critical point we have: local extrema, saddle point or not sure.

2. The two constraint theory

In max/min problems one often has a boundary. We dealt with this situation in two ways. First we parametrized the boundary curve and did one variable max/min theory on it. This still works. For the second method we introduced Lagrange multipliers in two variables. The corresponding method here is to require the boundary curve to be the intersection of two constraint surfaces, $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_2$.

If you parametrize the intersection curve by $\mathbf{r}(t)$ you need to compute $h'(t)$ where $h = f\big(\mathbf{r}(t)\big)$.

$$h'(t) = \nabla f\big(\mathbf{r}(t)\big) \bullet \mathbf{r}'(t)$$

Recall $\mathbf{r}'$ is parallel to $\nabla g_1 \times \nabla g_2$ so the triple scalar product vanishes at the critical points

$$(*) \qquad \det \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_1}{\partial z} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_2}{\partial z} \end{vmatrix} = 0$$

If we agree that the preferred direction on the curve is the direction given by $\nabla g_1 \times \nabla g_2$, then the determinant in $(*)$ is

- positive if f is increasing along the curve in the preferred direction
- negative if f is decreasing along the curve in the preferred direction
- 0 if we have a critical point of $f\big(\mathbf{r}'(t)\big)$.

If you have had more linear algebra you would know that this means either that

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

or that you are at a critical point of g_i which lies on $g_i(x, y, z) = c_i$. As in the one constraint case, in this course we will not pursue how to determine what sort of critical point we have: local extrema or not sure.

3. Some topology

Since we have no methods for analyzing what sort of critical points we have we need to fall back on the result that any continuous function on a closed bounded set has a minimum value and a maximum value and if the function is differentiable these must occur at critical points.

We will sometimes use a special result for distance functions, and hence squares of distance functions, which says that the distance function from a point to any set closed set has a minimum.

Hence to find the minimum value, write down a list which includes the critical points and plug in these into f. The smallest number you see is the minimum value. You can find the maximum value similarly.

Meditate on how dangerous this method is if your list does not contain all the critical points.

Here are some useful results to determine if you are working on a closed bounded set.

- Level sets of continuous functions are closed.
- The intersection of closed sets is closed.
- The union of a finite number of closed sets is closed.

These results have the practical consequence of implying that the sets that come up in Lagrange multipliers are closed.

- A subset of a bounded set is bounded.
- The intersection of bounded sets is bounded.
- The union of a finite number of bounded sets is bounded.
- A subset is bounded if and only if it is bounded in each coordinate.

Bounded is more problematic: $x^2 + y^2 = 1$ is bounded, $xy = 1$ is not. Lines and planes are never bounded.

4. Examples

Find the minimum distance from a point (x_0, y_0, z_0) to the plane $ax + by + cz + d = 0$. As usual we can minimize the distance squared so $f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ and $g(x, y, z) = ax + by + cz + d$ so the constriant surface is $g = 0$. $\nabla g = \langle a, b, c\rangle$ and $\nabla f = \langle 2(x - x_0), 2(y - y_0), 2(z - z_0)\rangle$ so

$$\begin{aligned} 2(x - x_0) &= \lambda a \\ 2(y - y_0) &= \lambda b \\ 2(z - z_0) &= \lambda c \\ ax + by + cz + d &= 0 \end{aligned}$$

$$\begin{aligned} x &= x_0 + \frac{\lambda}{2} a \\ y &= y_0 + \frac{\lambda}{2} b \\ z &= z_0 + \frac{\lambda}{2} c \\ ax + by + cz + d &= 0 \end{aligned}$$

so

$$\left(ax_0 + \frac{\lambda}{2} a^2\right) + \left(by_0 + \frac{\lambda}{2} b^2\right) + \left(cz_0 + \frac{\lambda}{2} c^2\right) + d = 0$$

or

$$ax_0 + by_0 + cz_0 + d + \frac{\lambda}{2}(a^2 + b^2 + c^2) = 0$$

or

$$\lambda = -\frac{2(ax_0 + by_0 + cz_0 + d)}{a^2 + b^2 + c^2}$$

Hence

$$\begin{aligned} x &= x_0 - a\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2} \\ y &= y_0 - b\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2} \\ z &= z_0 - c\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2} \end{aligned}$$

so at the critical point

$$f(x, y, z) = a^2\frac{(ax_0 + by_0 + cz_0 + d)^2}{(a^2 + b^2 + c^2)^2} + b^2\frac{(ax_0 + by_0 + cz_0 + d)^2}{(a^2 + b^2 + c^2)^2} + c^2\frac{(ax_0 + by_0 + cz_0 + d)^2}{(a^2 + b^2 + c^2)^2}$$

or

$$f(x, y, z) = \frac{(ax_0 + by_0 + cz_0 + d)^2}{a^2 + b^2 + c^2}$$

Hence the distance is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Maximize $f(x,y,z) = 4y - 2z$ subject to $g_1(x,y,z) = 2x - y - z = 2$ and $g_2(x,y,z) = x^2 + y^2 = 1$.

Since $x^2+y^2 = 1$, the x and y coordinates of the intersection curve are bounded. Since $2x-y-z = 2$, the z coordinate of the intersection curve is also bounded. Hence the intersection curve is closed and bounded.

- $\nabla f = \langle 0, 4, -2\rangle$
- $\nabla g_1 = \langle 2, -1, -1\rangle$
- $\nabla g_2 = \langle 2x, 2y, 0\rangle$
- $2x - y - z = 2$
- $x^2 + y^2 = 1$

Hence

$$\begin{aligned} 0 &= 2\lambda + 2\mu x \\ 4 &= -\lambda + 2\mu y \\ -2 &= -\lambda \\ 2x - y - z &= 2 \\ x^2 + y^2 &= 1 \end{aligned}$$

Hence $x = -\dfrac{2}{\mu}$, $y = \dfrac{3}{\mu}$.

Then $\dfrac{4}{\mu^2} + \dfrac{9}{\mu^2} = 1$ so $\mu = \pm\sqrt{13}$.

If $\mu = \sqrt{13}$,
$x = -\dfrac{2}{\sqrt{13}}$, $y = \dfrac{3}{\sqrt{13}}$ and finally $z = -2 - \dfrac{7}{\sqrt{13}}$

If $\mu = -\sqrt{13}$,
$x = \dfrac{2}{\sqrt{13}}$, $y = -\dfrac{3}{\sqrt{13}}$ and finally $z = -2 + \dfrac{7}{\sqrt{13}}$

$$\begin{aligned} f\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2 - \frac{7}{\sqrt{13}}\right) &= 4 + \frac{26}{\sqrt{13}} \\ f\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2 + \frac{7}{\sqrt{13}}\right) &= 4 - \frac{26}{\sqrt{13}} \end{aligned}$$

I do not get full marks for the exposition above!

Before you scroll down to the next page you might want to think about why not.

The problem comes when I wrote $x = -\dfrac{2}{\mu}$, $y = \dfrac{3}{\mu}$. This assumes $\mu \neq 0$ and if there are critical points with $\mu = 0$ I may have lost them. This is germane to the meditation in §3.

However, I did get the right answer this time since if $\mu = 0$, $0 = 2\lambda + 2\mu x$ implies $\lambda = 0$ and we already know $\lambda = 2$. This contradiction implies $\mu \neq 0$.