1. Length

Let $\mathbf{r}(t)$ be a parametrization of a piece of a curve C.

$$D = \int_a^b |\mathbf{r}'(t)|\, dt$$

is the *distance travelled* by the particle $a \leqslant t \leqslant b$. If the particle goes from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ without turning around then D is the length of the curve between those two points. If $|\mathbf{r}'(t)| > 0$ for $a \leqslant t \leqslant b$ then the particle is unable to turn around, at least if $|\mathbf{r}'(t)|$ is continuous. A parametrization is *smooth* if $\mathbf{r}'$ is continuous and never the zero vector. A particle moving with a smooth parametrization can never turn around.

2. Arc length parametrization

As long as the curve C has some differential parametrization, $\mathbf{r}(t)$, the distance between two points on the curve is well defined. Pick an initial point on the curve, say $\mathbf{p}_0 = \mathbf{r}(t_0)$. Let ω be the least upper bound of the distance from $\mathbf{p}_0$ to any point on the curve moving in the direction of increasing t. Let α be the least upper bound of the distance from $\mathbf{p}_0$ to any point going in the direction of decreasing t. Both α and ω may be $+\infty$, for example they both are for a line or a circle, or indeed for many curves. A line segment is an example with α and ω finite.

Define a parametrization of the curve, called an *arc length parametrization*, as follows. For each $s \in (-\alpha, \omega)$ define $\mathbf{p}(s)$ to be the point on the curve a distance s from $\mathbf{p}_0$ on the preferred half if $s \geqslant 0$ or a distance $-s$ from $\mathbf{p}_0$ on the other half if $s \leqslant 0$.

It can be proved that $\mathbf{p}(s)$ is a differentiable function of s so $\displaystyle\int_0^s |\mathbf{r}'(t)|\, dt$ is the distance from $\mathbf{p}_0$ to $\mathbf{p}(s)$, which by definition is $|s|$. Differentiate $|s| = \displaystyle\int_0^s |\mathbf{r}'(t)|\, dt$ using the Fundamental Theorem of calculus to see that $|\mathbf{r}'(s)| = 1$ so

$$\mathbf{T}(s) = \mathbf{p}'(s)$$

Given any smooth parametrization $\mathbf{r}(t)$ there is a function $f(t)$ so that

$$\mathbf{r}(s) = \mathbf{p}\big(f(t)\big)$$

where $f(t)$ is the signed distance from $\mathbf{p}_0$ to $\mathbf{r}(t)$. Then $|\mathbf{r}'(t)| = |f'(t)|\big|\mathbf{p}'\big(f(t)\big)\big| = |f'(t)|$. Hence if $|\mathbf{r}'(t)| = 1$, $f'(t) = \pm 1$ and $f(t) = \pm t + c$. If the sign is minus, the $\mathbf{r}$ parametrization chooses the other half for the preferred side. Various choices for c change the initial point.

A parametrization $\mathbf{r}(t)$ with $|\mathbf{r}'(t)| = 1$ $t \in (-\alpha, \omega)$ is an arc length parametrization with initial point $\mathbf{r}(0)$ which moves into the preferred half of the curve for positive t.

Example: Let L be the line $\mathbf{r}(t) = \mathbf{q} + t\mathbf{v}$ for $t \in (-\infty, \infty)$. (So $\mathbf{v} \neq \mathbf{0}$.) Note $\mathbf{q} + \mathbf{v}$ is in the preferred half. Then

$$\mathbf{p}(s) = \mathbf{q} + s\frac{\mathbf{v}}{|\,\mathbf{v}\,|} \qquad -\infty < s < \infty$$

The distance from $\mathbf{p}(s_0)$ to $\mathbf{p}(s_1)$ is $|\mathbf{p}(s_1) - \mathbf{p}(s_0)|$ and you can calculate that this is $|s_1 - s_0|$ as it is supposed to be. Note also that $\mathbf{r}(t) = \mathbf{p}\big(|v| \cdot s\big)$

Example: Let C be the circle of radius r centered at the origin in the plane, so $\mathbf{r}(t) = \langle r\cos(t), r\sin(t)\rangle$ $t \in [0, 2\pi)$ is a parametrization. In this case

$$\mathbf{p}(s) = \left\langle r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\right\rangle \qquad 0 \leqslant s < 2\pi r$$

is an arc length parametrization. Note $\mathbf{T}(s) = \left\langle -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right\rangle$.

Very important point: In the next section we will define several interesting and important concepts using arc length parametrization. These should be thought of as definitions and you should try to understand what they are trying to tell you about the curve. BUT, in the next lecture we will discuss useful methods of calculation. There will be a shortage of examples for now because for most curves it is impossible to work out the arc length parametrization explicitly.

3. Curvature & related concepts

How does $\mathbf{T}$ change as you move along the curve? Or in other words, what is $\frac{d\mathbf{T}}{ds} = \mathbf{T}'$?

Since $\mathbf{T}\bullet\mathbf{T} = 1$, $\mathbf{T}\bullet\mathbf{T}' = 0$. Define the *principal normal vector* to the curve as $\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|}$ and define the *curvature* of the curve at the point $\mathbf{r}(s)$ to be $\kappa(s) = |\mathbf{T}'(s)|$ so that

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$

If $\mathbf{T}'(s)$ is the zero vector, $\mathbf{N}$ is not defined but we will define $\kappa(s) = 0$. Note that always $\kappa(s) \geqslant 0$.

When $\mathbf{N}$ is defined, $\mathbf{T}$ and $\mathbf{N}$ are orthogonal. The tangent line to the curve at the point s is $\mathbf{p}(s) + t\mathbf{T}(s)$ so it lies in the plane spanned by $\mathbf{T}(s)$ and $\mathbf{N}(s)$. The projection of the curve into this plane lies on the side of the tangent line pointed to by $\mathbf{N}(s)$ and $\kappa(s)$ measures how fast it is receding from the tangent line.

For the line, $\mathbf{N}$ is not defined and so lines have curvature 0. For the circle,

$$\mathbf{T}'(s) = \frac{1}{r}\left\langle -\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right)\right\rangle$$

so

$$\mathbf{N}(s) = \left\langle -\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right)\right\rangle$$

and $\kappa(s) = \frac{1}{r}$ is a constant which depends only on the radius. Sometimes you will see $\frac{1}{\kappa(s)}$ referred to as the *radius of curvature*.

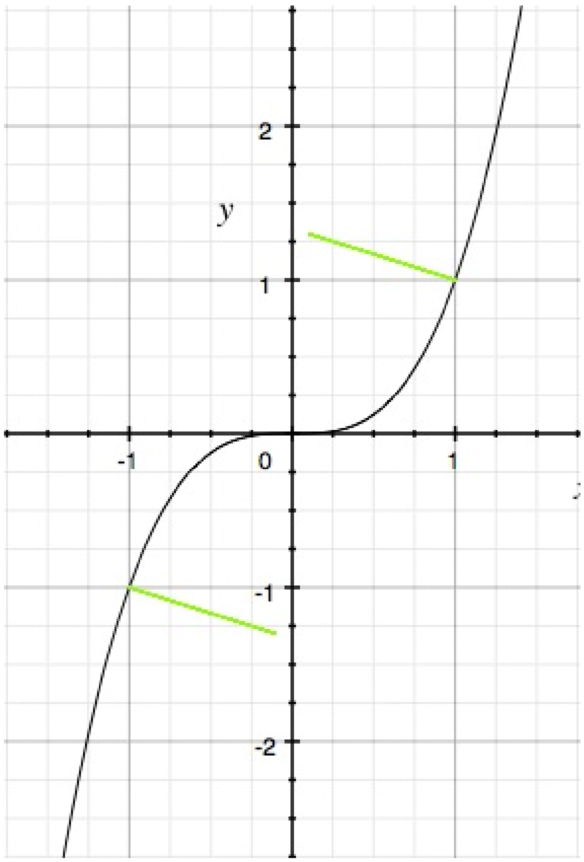
Since $\mathbf{N}\bullet\mathbf{N} = 1$, $\mathbf{N}'\bullet\mathbf{N} = 0$. Since $\mathbf{N}\bullet\mathbf{T} = 0$, $\mathbf{N}'\bullet\mathbf{T} + \mathbf{N}\bullet\mathbf{T}' = \mathbf{N}'\bullet\mathbf{T} + \kappa(s)\mathbf{N}\bullet\mathbf{N} = 0$, so $\mathbf{N}'\bullet\mathbf{T} = -\kappa$.

$$\begin{aligned} \mathbf{N}'\bullet\mathbf{N} &= 0 \\ \mathbf{N}'\bullet\mathbf{T} &= -\kappa \end{aligned} \tag{1}$$

3.1. **The plane.** In the plane, every vector can be written uniquely as a linear combination of $\mathbf{T}(s)$ and $\mathbf{N}(s)$, assuming of course that $\mathbf{N}(s)$ exists. Then we have $\mathbf{N}' = a\mathbf{T} + b\mathbf{N}$ and from (1)

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}$$

Even for very pleasant curves, $\mathbf{N}$ can be discontinuous. Near a point on the curve, the curve divides the plane into two pieces, as does the tangent line to the curve at that point. If the curve has an $\mathbf{N}$, then it picks out one side or the other and the curve lies on that side of the tangent line. Hence at an inflection point of the curve the principal normal is undefined. It jumps from one side to the other nearby.



Here is a piece of the graph of $y = x^3$ which recall has an inflection point at the origin. The two green lines are $\mathbf{N}$ at $x = \pm 1$. Notice in both cases the curve lies on one side of the tangent line. You probably called the curve x concave down at $x = -1$ and concave up at $x = 1$. More to the point however, there is no way to move the unit length green vectors towards the origin continuously, one of them arrives pointing straight up and the other straight down.

3.2. **Three space.** Assuming $\mathbf{N}$ exists, define the *binormal* vector to the curve at the point s to be

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$$

Then any vector in three space can be written uniquely as $a\mathbf{T} + b\mathbf{N} + c\mathbf{B}$. Hence

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

where $\tau(s)$ is a function called the *torsion* of the curve at the point.

Since $\mathbf{B}' = \mathbf{T}' \times \mathbf{N} + \mathbf{T} \times \mathbf{N}'$, $\mathbf{B}' = -\tau\mathbf{N}$ we obtain the *Frenet-Serret formulas*

$$\begin{aligned}
\mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s) \\
\mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s)
\end{aligned}$$

The plane spanned by the tangent and the principal normal is called the *osculating plane*. At the point $\mathbf{r}(s_0)$ the parametric equation is

$$\mathbf{r}(s_0) + x\mathbf{T}(s_0) + y\mathbf{N}(s_0)$$

The vector equation is

$$\mathbf{B}(s_0) \bullet \langle x, y, z\rangle = \mathbf{B}(s_0) \bullet \mathbf{r}(s_0)$$

If the curve lies in a plane, this is that plane. A normal vector to the osculating plane is the binormal and the torsion measures the desire of the osculating plane to twist.

The *osculating circle* at a point $\mathbf{r}(s_0)$ is the circle in the osculating plane at $\mathbf{r}(s_0)$ with the same curvature and tangent. It turns out that the center of the osculating circle lies on $\mathbf{N}(s_0)$ at a distance $1/\kappa(s_0)$ from the initial point so the center of the osculating circle is at $\mathbf{r}(s_0) + \dfrac{1}{\kappa(s_0)}\mathbf{N}(s_0)$.

The *normal plane* to the curve at the point $\mathbf{r}(s_0)$ is the plane through $\mathbf{r}(s_0)$ perpendicular to the curve. Its equation is

$$\mathbf{T}(s_0) \bullet \langle x, y, z\rangle = \mathbf{T}(s_0) \bullet \mathbf{r}(s_0)$$

It is also the plane spanned by $\mathbf{N}$ and $\mathbf{B}$ and so has parametric equation

$$\mathbf{r}(s_0) + x\mathbf{N}(s_0) + y\mathbf{B}(s_0)$$

You will be pleased to learn that no one seems to have named the plane at a point on the curve spanned by the tangent and the binormal.

The numbers $\kappa(s)$ and $\tau(s)$ are *intrinsic* to the curve as is the vector $\mathbf{N}$. This means that given a point on the curve, it makes sense to ask for the curvature and the torsion of the curve at that point as well as for the normal vector at that point. The tangent and binormal vectors depend on the oriented curve.

Example: The curve $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ $-\infty < t < \infty$ is a helix. Assume a and b are positive and let $c = \sqrt{a^2+b^2}$. Consider the curve

$$\mathbf{p}(s) = \left\langle a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right\rangle \qquad -\infty < s < \infty$$

Then

$$\mathbf{p}'(s) = \left\langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right\rangle$$

and $|\mathbf{p}'(s)| = 1$ so $\mathbf{p}$ is a curve parametrized by arc length. Note $\mathbf{r}(t) = \mathbf{p}(c\,s)$.

Since

$$\mathbf{T}(s) = \left\langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right\rangle$$

we have

$$\mathbf{T}'(s) = \left\langle -\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right\rangle$$

Hence

$$\mathbf{N}(s) = \left\langle -\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right\rangle$$

and

$$\kappa(s) = \frac{a}{c^2}$$

$$\mathbf{B}(s) = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right) & -\sin\left(\frac{s}{c}\right) & 0 \end{vmatrix} = \left\langle \frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right\rangle$$

and

$$\mathbf{B}'(s) = \left\langle \frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right\rangle$$

so

$$\tau(s) = \frac{b}{c^2}$$

Note

$$\mathbf{N}'(s) = \left\langle \frac{1}{c}\sin\left(\frac{s}{c}\right), -\frac{1}{c}\cos\left(\frac{s}{c}\right), 0\right\rangle =$$

$$-\frac{a}{c^2}\left\langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right\rangle + \frac{b}{c^2}\left\langle \frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right\rangle =$$

$$-\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

Hence $\kappa(s) = \dfrac{a}{c^2}$ and $\tau(s) = \dfrac{b}{c^2}$. Here is an animation of the Frenet-Serret frame for a helix.

3.3. **Vanishing torsion.** As I remarked in class, a curve lies in a plane if and only if the torsion is 0 at all points on the curve.

To see this, consider the function $g(s) = \mathbf{B}(s) \bullet \mathbf{r}(s)$. Then $g'(s) = \mathbf{B}'(s)\bullet\mathbf{r}(s) + \mathbf{B}(s)\bullet\mathbf{r}'(s) = -\tau(s)\mathbf{N}\bullet\mathbf{r}(s) + \mathbf{B}(s)\bullet\mathbf{r}'(s) = -\tau(s)\mathbf{N}\bullet\mathbf{r}(s) + \mathbf{B}(s)\bullet\kappa(s)\mathbf{T}(s)$. Hence, if $\tau(s) = 0$, $g'(s) = 0$ and $g(s) = \mathbf{B}(s)\bullet\mathbf{r}(s) = c$ for some constant c.

Moreover, $\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$ so if $\tau(s) = 0$, $\mathbf{B}'(s) = \mathbf{0}$ and so $\mathbf{B}$ is constant, say $\mathbf{B}(s_0)$. Putting these two equations together, if $\tau(s) = 0$, $\mathbf{B}(s_0)\bullet\mathbf{r}(s) = c$ for all s and the curve lies in a plane which is the osculating plane at any point along the curve.

Conversely, if the curve lies in a plane, then there is a unit vector $\mathbf{D}$ and a constant c such that $\mathbf{D}\bullet\mathbf{r}(s) = c$. But then $\mathbf{D}\bullet\mathbf{r}'(s) = \mathbf{D}\bullet\mathbf{T}(s) = 0$. Differentiating again $\mathbf{D}\bullet\mathbf{T}'(s) = \kappa(s)\mathbf{D}\bullet\mathbf{N}(s) = 0$, so $\mathbf{D}$ is perpendicular to $\mathbf{T}$ and $\mathbf{N}$ and hence parallel to $\mathbf{B}(s)$ and hence $\mathbf{D} = \pm\mathbf{B}(s)$ for all s. Differentiating one more time $\mathbf{0} = \mp\tau(s)\mathbf{N}(s)$ so $\tau(s) = 0$.

The above argument fails if there is no $\mathbf{N}$ or equivalently if $\kappa(s) = 0$ for all s. But $\kappa(s) = 0$ implies $\mathbf{r}(s)$ is a line and hence also lies in a plane, in fact in lots of planes.

To see $\kappa(s) = 0$ implies the that the curve is a line, note first that $\kappa(s) = 0$ implies that $\mathbf{T}'(s) = \mathbf{0}$ or $\mathbf{T}(s)$ is constant and so $\mathbf{T}(s) = \mathbf{T}(0)$. Let $\mathbf{q}(s) = \mathbf{r}(0) + s\mathbf{T}(0)$ be a line and consider $\mathbf{w}(s) = \mathbf{r}(s) - \mathbf{q}(s)$. Then $\mathbf{w}'(s) = \mathbf{r}'(s) - \mathbf{T}(0) = \mathbf{T}(s) - \mathbf{T}(0) = \mathbf{0}$ so $\mathbf{w}(s)$ is a constant and since $\mathbf{w}(0) = \mathbf{0}$,

$$\mathbf{r}(s) = \mathbf{r}(0) + s\mathbf{T}(0)$$