Suppose given a curve C in the plane or in space and a function F defined on the curve whose output is real numbers. Then we can define the *line integral of f along C* as usual via Riemann sums. Cut C up into lots of short pieces and pick a point in each little piece. For each piece, evaluate the function f at that point and multiply the answer by the length of the little piece. Add up all these answers. Take the limit as the pieces get shorter and shorter.

$$\int_C f\,ds = \lim \sum_{i=1}^{n} F(p_i^*) d(p_i, p_{i-1})$$

where the curve is cut at points $p_0 = \textit{the start}$, $p_1, \ldots, p_n = \textit{the end}$ and where p_i^* is a point on the curve between p_{i-1} and p_i. The term $d(p_i, p_{i-1})$ is the distance between p_{i-1} and p_i using the distance formula and is an approximation to the actual length of the curve between these two points. The limit is taken over partitions with all the $d(p_i, p_{i-1})$ getting smaller and smaller.

Recall that a smooth parametrization of a curve is a function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$ so that the curve starts at $\mathbf{r}(a)$, ends at $\mathbf{r}(b)$, and $\mathbf{r}'(t) \neq \mathbf{0}$. It is a theorem, essentially due to Riemann, that if F is continuous on C and C has a smooth parametrization, then $\int_C F\,ds$ exists.

From our work on curves it is easy convert the line integral into an ordinary integral:

$$\int_C F\,ds = \int_a^b F\big(\mathbf{r}(t)\big) \cdot |\mathbf{r}'(t)|\,dt \tag{1}$$

This is the formula in 2 dimensions, 3 dimensions, or any other dimension in which you might be interested.

By definition, $\int_C F\,ds$ does not depend on any parametrization. Use substitution in one variable to show that if you change parametrization for your curve, $\int_a^b F\big(\mathbf{r}(t)\big) \cdot |\mathbf{r}'(t)|\,dt$ stays the same provided your parametrization is smooth and starts]and ends at the same points as the original parametrization.

1. Properties

If $F \leqslant G$ along C then

$$\int_C F\,ds \leqslant \int_C G\,ds \tag{P1}$$

with equality if and only if $F = G$.

$$\int_C (F+G)\,ds = \int_C F\,ds + \int_C G\,ds \tag{P2}$$

$$\int_C c \cdot F\,ds = c\int_C F\,ds \tag{P3}$$

If we write $C = C_1 \cup C_2$ where $C_1 \cap C_2$ is one or two points,

$$\int_C F\,ds = \int_{C_1} F\,ds + \int_{C_2} F\,ds \tag{P4}$$

2. Review of parametrized curves

(1) $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ or $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leqslant t \leqslant b$.
(2) Graph $y = f(x)$, $a \leqslant x \leqslant b$: $\mathbf{r}(t) = \langle t, f(t)\rangle$, $a \leqslant t \leqslant b$.
(3) Polar graph $r = f(\theta)$, $a \leqslant \theta \leqslant b$: $\mathbf{r}(t) = \langle f(t)\cos t, f(t)\sin t\rangle$, $a \leqslant t \leqslant b$.

$$\int_C F\, ds = \int_a^b F\big(\mathbf{r}(t)\big) \cdot |\mathbf{r}'(t)|\; dt$$

(1) $|\mathbf{r}'(t)| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}$ or $|\mathbf{r}'(t)| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$
$F(x, y) = F\big(x(t), y(t)\big)$ or $F(x, y, z) = F\big(x(t), y(t), z(t)\big)$.
(2) Graph $y = f(x)$, $|\mathbf{r}'(t)| = \sqrt{1 + \big(f'(t)\big)^2}$.
$F(x, y) = F\big(t), f(t)\big)$.
(3) Polar graph $r = f(\theta)$, $|\mathbf{r}'(t)| = \sqrt{\big(f(t)\big)^2 + \big(f'(t)\big)^2}$.
$F(x, y) = F\big(f(t)\cos t, f(t)\sin t\big)$.

The first formula for $|\mathbf{r}'(t)|$ is the definition. Formula (3) is worth remembering because the derivation of it from formula (1) is straightforward but you need to get signs right and be careful with the algebra.

Later in the semester we will discuss how to deal with level curves but for now you need to parametrize the curve to make progress.

3. Applications

The standard list:

- Length of a curve C: $\int_C 1\,ds$.
- Mass of a curve C with variable density μ: $\int_C \mu\,ds$.
- Charge of a curve C with charge density μ: $\int_C \mu\,ds$.
- Moments and centers of mass: $\bar{x} = \dfrac{\int_C x\mu\,ds}{\int_C \mu\,ds}$, $\bar{y} = \dfrac{\int_C y\mu\,ds}{\int_C \mu\,ds}$ and (if needed) $\bar{z} = \dfrac{\int_C z\mu\,ds}{\int_C \mu\,ds}$.
- Moments of inertia:

  2-dimensions

  - $I_x = \int_C x^2\mu\,ds$,
  - $I_y = \int_C y^2\mu\,ds$
  - Moment of inertia about the origin $= I_x + I_y$

  3-dimensions

  - $I_x = \int_C y^2 + z^2\mu\,ds$,
  - $I_y = \int_C x^2 + z^2\mu\,ds$
  - $I_z = \int_C x^2 + y^2\mu\,ds$

  Radius of gyration: $R_x^2 = \dfrac{I_x}{\text{mass}}$, $R_y^2 = \dfrac{I_y}{\text{mass}}$ and if needed $R_z^2 = \dfrac{I_z}{\text{mass}}$.
- Area of a fence along the curve C is h is the height of the fence: $\int_C h\,ds$.