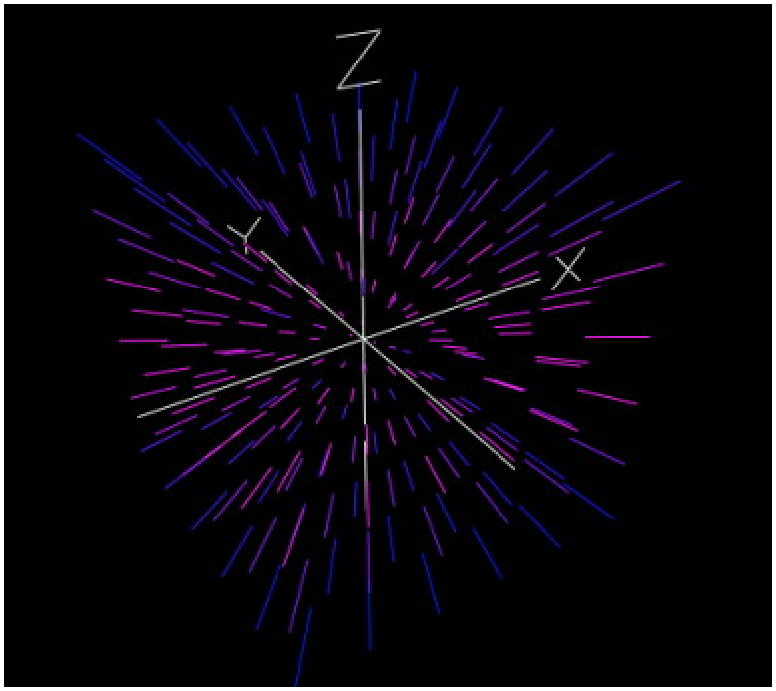
1. Vector fields

A major use of line integrals involves integrating a curve along a vector field, which sort of presupposes you know what a vector field is. Fortunately you do: a *vector field* is just a function which takes as inputs a vector and returns a vector of the same type. It's kind of like a change of coordinates except there is no one-to-one condition you need to worry about and the point of view is completely different.

A vector field $\mathbf{v}$ is usually thought of as follows. Start with a point in the domain of $\mathbf{v}$ and evaluate $\mathbf{v}$ at that point: say (a, b) and $\mathbf{v}(a, b)$. Then the vector field at (a, b) is the vector $\mathbf{v}(a, b)$ starting at (a, b). Here is one picture.

From http://library.oceanteacher.org/OTMediawiki/images/b/b9/ResultantVectors.jpg

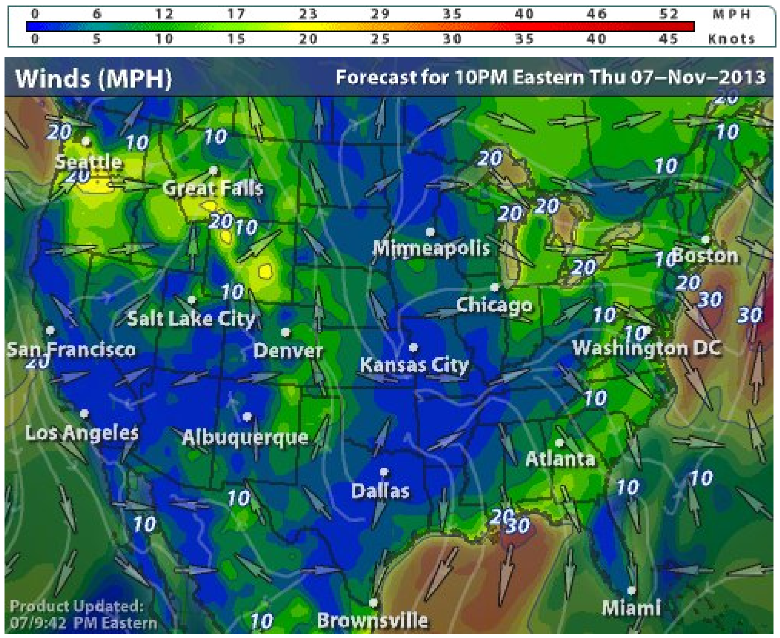
Here is a 3-dimensional one.



From http://www.mathstudioapp.com/share/images/1331/preview189.png

Drawing a vector field is an art form like drawing level curves. Every point has an associated vector but if you draw them all the picture will be too crowded to see much whereas if you draw too few you may miss important features. Still, pick some points, evaluate the vector field at those points and then draw each vector starting at the point you plugged in. Easier is to just use some graphing software, most of which have a vector field option.

Here is a picture which shows one reason vector fields come up.



From http://www.intellicast.com/National/Wind/WINDcast.aspx

The information is presented to you in a slightly different manner. All the vectors are unit vectors pointing in the direction in which the wind is blowing. The speed is indicated by color rather than the length of the vector.

Some important examples are

- Gravitational force field for a particle of mass M: if the particle is at the origin
  $\mathbf{v} = \dfrac{GM}{|\mathbf{r}|^3}\mathbf{r}$
- Electrical force field for a particle of charge Q: if the particle is at the origin
  $\mathbf{v} = \dfrac{Q}{4\pi\epsilon_0\,|\mathbf{r}|^3}\mathbf{r}$
- Fluid flow
- Strong nuclear force, weak nuclear force, Star Trek force fields

  $\vdots$

Since forces add and vectors add, if you need to know the force field for several forces, add the vector fields for each one.

An important concept associated to a vector field $\mathbf{v}$ are the *flow lines.* A curve $\mathbf{r}$ is a flow line for a vector field $\mathbf{v}$ provided $\mathbf{r}'(t)$ and $\mathbf{v}\big(\mathbf{r}(t)\big)$ are parallel for all t. A flow line comes with a preferred direction, namely the direction the direction in which $\mathbf{v}\big(\mathbf{r}(t)\big)$ points.

The name comes from the fact that if you drop a particle into a stream, or put a balloon into the air, it will travel along a flow line. A similar result holds for a particle moving in a force field.

Example. Let $\mathbf{v} = \langle x, y\rangle$ so this is a force directed along the rays out of the origin and the further out you are the faster you are going. The curves $y = mx$ and $x = 0$ are flow lines:

$\mathbf{r}(t) = \langle t, mt\rangle$, $\mathbf{r}'(t) = \langle 1, m\rangle$ and $\mathbf{v}\big(\mathbf{r}(t)\big) = \langle t, mt\rangle = \langle 1, m\rangle\, t$ so $\mathbf{r}'(t)$ and $\mathbf{v}\big(\mathbf{r}(t)\big)$ are parallel

An important class of examples of vector fields are the *conservative* ones. A field $\mathbf{v}$ is *conservative* provided $\mathbf{v} = \nabla p$ for some function p which is called a *potential* function for $\mathbf{v}$. Given one potential function p for $\mathbf{v}$, any other potential function has the form $p + c$ for c a constant and $p + c$ is a potential function for $\mathbf{v}$ for any constant c. As usual, the constant can be determined if you know a value for the potential function at a particular point.

Flow lines for a conservative field just follow the gradient of a potential function.

2. Line integrals with vector fields

An important class of line integrals integrates $F = \mathbf{v} \bullet \mathbf{T}$ along the curve where T is the unit tangent vector. One example of this is *work*. If a particle is moving along a straight line in a constant force field, then the work done is just the dot product of the force and the unit tangent vector to the line, times the length of the line. A Riemann sum argument shows that in general,

$$Work = \int_C \mathbf{v} \bullet \mathbf{T}\, ds$$

These special integrals are typically easier to do than general ones, even though it looks like it will be more complicated. The set up initially is

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_a^b \mathbf{v}\big(\mathbf{r}(t)\big) \bullet \mathbf{T}\big(\mathbf{r}(t)\big)\, |\mathbf{r}'(t)|\ dt$$

But do not do this! Instead, recall

$$\mathbf{T}\big(\mathbf{r}(t)\big) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

so the setup becomes

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_a^b \mathbf{v}\big(\mathbf{r}(t)\big) \bullet \mathbf{r}'(t)\, dt \tag{2}$$

We often write $d\mathbf{r} = \mathbf{r}'(t)\, dt$ and then

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_C \mathbf{v} \bullet d\mathbf{r} = \int_a^b \mathbf{v}\big(\mathbf{r}(t)\big) \bullet d\mathbf{r}(t) \tag{2a}$$

Here is an example of how to do the required calculations. Let $\mathbf{v} = \langle yz, xz, xy\rangle$ and $\mathbf{r} = \langle \cos t, \sin t, t\rangle$, $0 \leqslant t \leqslant 2\pi$. Then $d\mathbf{r} = \langle -\sin t, \cos t, 1\rangle\ dt$ so

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_0^{2\pi} \langle t\sin t, t\cos t, \cos t \sin t\rangle \ \bullet\ \langle -\sin t, \cos t, 1\rangle\ dt =$$

$$\int_0^{2\pi} t(\cos^2 t - \sin^2 t) + \cos t \sin t\, dt = \int_0^{2\pi} t(\cos 2t)\, dt + \frac{1}{2}\sin^2 t\Big|_0^{2\pi} = \int_0^{2\pi} t(\cos 2t)\, dt =$$

$$\frac{1}{4}\int_0^{4\pi} u\cos u\, du = u\sin u + \cos u\Big|_0^{4\pi} = 0$$

In the plane there is another integral important for applications

$$\int_C \mathbf{v} \bullet \mathbf{N}\, ds$$

where N is a unit normal to the curve. If $\mathbf{v}$ is the velocity field for a fluid, $\int_C \mathbf{v} \bullet \mathbf{N}\, ds$ measures the amount of fluid crossing the curve. As an example, if $\mathbf{v}$ is the velocity field for a fluid such as water, which is incompressible and if C is a closed curve, $\int_C \mathbf{v} \bullet \mathbf{N}\, ds = 0$.

Just as in the tangential case $\int_C \mathbf{v} \bullet \mathbf{N}\, ds$ looks like it might be hard to set up. But note if $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$ then $\langle -y'(t), x'(t)\rangle$ is a normal vector. Since $|\mathbf{r}'(t)| = |\langle -y'(t), x'(t)\rangle|$,

$$\int_C \mathbf{v} \bullet \mathbf{N}\, ds = \int_a^b \mathbf{v}\big(\mathbf{r}(t)\big) \bullet \langle -y'(t), x'(t)\rangle)\, dt$$

If we write $d\mathbf{n} = \langle -y'(t), x'(t)\rangle)\, dt$,

$$\int_C \mathbf{v} \bullet \mathbf{N}\, ds = \int_C \mathbf{v} \bullet d\mathbf{n}$$

We could have used $\langle y'(t), -x'(t)\rangle$ as our normal vector. The choice we made was so as to have the following property.

If you walk along the curve in the direction of the tangent vector, the normal vector points to your left.

As an example, if you walk counterclockwise around the unit circle, the normal vector $< -\cos(\theta), -\sin(\theta) >$ points to your left which is into the disk.

3. Alternate notations

In addition to the notations above you will often see

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_C M(x,y)dx + N(x,y)dy$$

or

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$$

Here

- $\mathbf{v} = \langle M(x,y), N(x,y)\rangle$ and $\mathbf{T}ds = \langle x'(t), y'(t)\rangle\, dt$ so $dx = x'(t)dt$; $dy = y'(t)dt$ if $\mathbf{r}(t) = \langle x(t), y(t)\rangle$
- $\mathbf{v} = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$ and $\mathbf{T}ds = \langle x'(t), y'(t), z'(t)\rangle\, dt$ if $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

There are also arc length integral variations:

- $\int_C F(x,y)\, dx = \int_a^b F\big(x(t), y(t)\big)x'(t)\, dt$
- $\int_C F(x,y)\, dy = \int_a^b F\big(x(t), y(t)\big)y'(t)\, dt$

and similar integrals in 3 dimensions.

There is not so much to remember if you organize yourself correctly. There is always a curve, C which you need to parametrize so $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ and maybe a $z(t)$. In any line integral you happen to see, replace x by $x(t)$, y by $y(t)$ and z by $z(t)$ and then replace dx by $x'(t)\, dt$, dy by $y'(t)\, dt$ and dz by $z'(t)\, dt$. If you see ds replace it by $|\mathbf{r}'(t)|\, dt$. If you see $d\mathbf{r}$ replace it by $\mathbf{r}'(t)\, dt$. If you see $\mathbf{N}\, ds$ replace it by $\langle -y'(t), x'(t)\rangle\, dt$.

A smooth parametrization of a curve yields an *orientation* for the curve. An orientation is just a choice of direction to move along the curve at each point. Since the parametrization is smooth this choice will be continuous along the entire curve and so you can indicate your choice by just putting a little arrow at enough points along the curve. “Enough” here means that you can figure out which way you should be going at any point on the curve. For example see Figure 1 below.

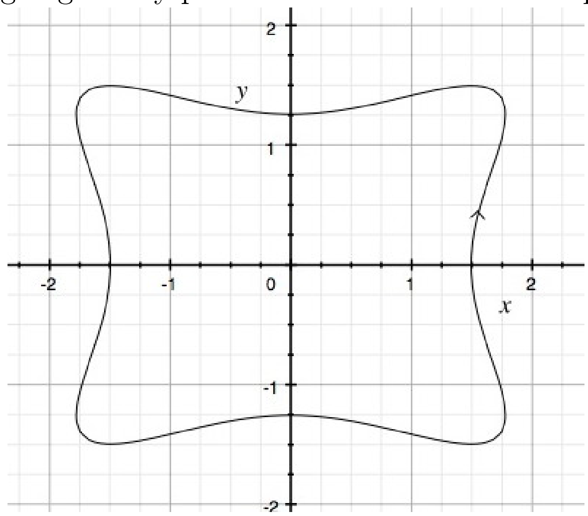


Figure 1

Some times, more than one arrow is needed to indicate exactly what is going on

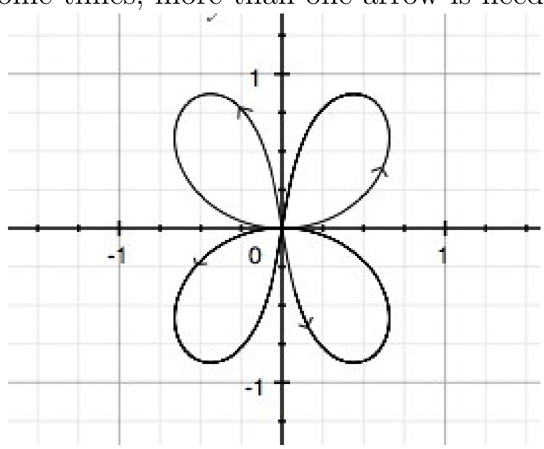


Figure 2

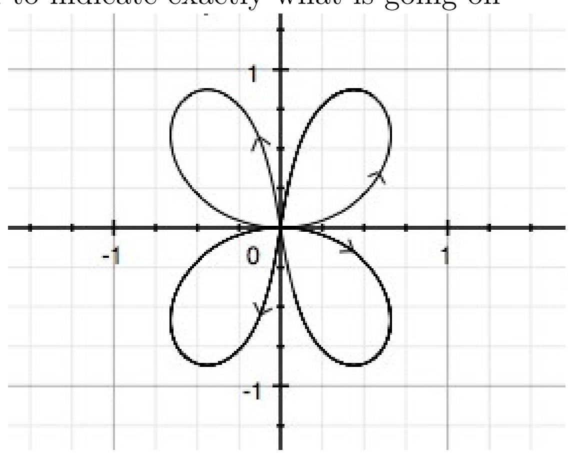


Figure 3

Here we have indicated two ways a particle might move along the curve and unless you see formulas you have no way of knowing which is what was intended.

If the curve has more than one piece, a minimum of one arrow on each piece is required.

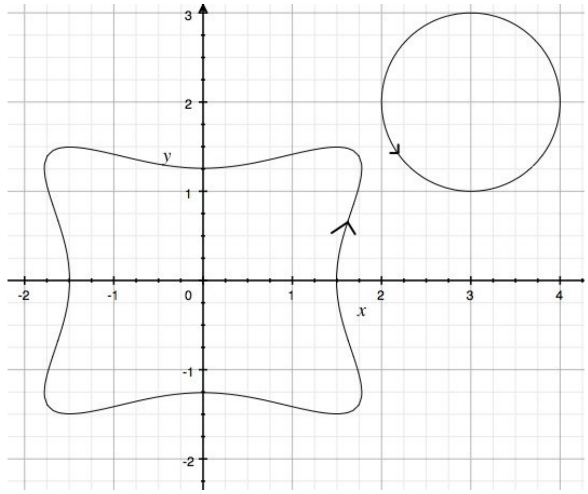


Figure 4

The relation of orientation to line integrals is the following. $\int_C F\,ds$ is independent of orientation but $\int_C \mathbf{v} \bullet \mathbf{T}\,ds$ changes sign if you reverse the orientation. In the plane, $\int_C \mathbf{v} \bullet \mathbf{N}\,ds$ also changes sign if you reverse the orientation.

Given an oriented curve in the plane, the preferred normal vector points to your left as you move along the curve in the preferred direction.