1. The fundamental theorem of line integrals

In general $\int_C \mathbf{v} \bullet \mathbf{T}\, ds$ depends on the curve between the starting point and the ending point. Consider two ways to get from $(1,0)$ to $(0,1)$:

(1) C_1: start at $(1,0)$ and first go along the x-axis to the origin; then go up the y-axis to $(0,1)$.
(2) C_2: start at $(1,0)$ and first go straight up to $(1,1)$; then go parallel to the x-axis to $(0,1)$.

Suppose $\mathbf{v} = \langle y, 2x\rangle$. To parametrize things easily, let us do four line integrals along lines: L_1: $(0,0)$ to $(1,0)$; L_2: $(0,0)$ to $(0,1)$; L_3: $(0,1)$ to $(1,1)$ and L_4: $(1,0)$ to $(1,1)$. All t intervals are $[0,1]$.

L_1: $\mathbf{r}(t) = \langle t, 0\rangle$; $\mathbf{r}'(t) = \langle 1, 0\rangle\ dt$; $\mathbf{v}\big(\mathbf{r}(t)\big) = \langle 0, 2t\rangle$ so $\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_0^1 0\, dt = 0$.

L_2: $\mathbf{r}(t) = \langle 0, t\rangle$; $\mathbf{r}'(t) = \langle 0, 1\rangle\ dt$; $\mathbf{v}\big(\mathbf{r}(t)\big) = \langle t, 0\rangle$ so $\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_0^1 0\, dt = 0$.

L_3: $\mathbf{r}(t) = \langle t, 1\rangle$; $\mathbf{r}'(t) = \langle 1, 0\rangle\ dt$; $\mathbf{v}\big(\mathbf{r}(t)\big) = \langle 1, 2t\rangle$ so $\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_0^1 1\, dt = 1$.

L_4: $\mathbf{r}(t) = \langle 1, t\rangle$; $\mathbf{r}'(t) = \langle 0, 1\rangle\ dt$; $\mathbf{v}\big(\mathbf{r}(t)\big) = \langle t, 2\rangle$ so $\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_0^1 2\, dt = 2$.

Then

$$\int_{C_1} \mathbf{v} \bullet \mathbf{T}\, ds = -\int_{L_1} \mathbf{v} \bullet \mathbf{T}\, ds + \int_{L_2} \mathbf{v} \bullet \mathbf{T}\, ds = 0$$
$$\int_{C_2} \mathbf{v} \bullet \mathbf{T}\, ds = \int_{L_4} \mathbf{v} \bullet \mathbf{T}\, ds - \int_{L_3} \mathbf{v} \bullet \mathbf{T}\, ds = 2 - 1 = 1$$

Be sure you understand these equations. The figure below should explain the various signs.

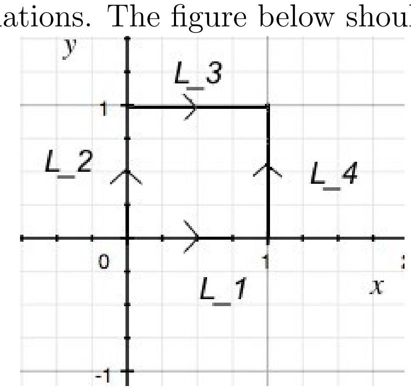


Figure 5: Some oriented lines

The four lines L_i are labeled and arrows have been added to show the orientations induced by the parametrizations given above. Then $C_1 = (-L_1) \cup (L_2)$ and $C_2 = (L_4) \cup (-L_3)$ as oriented curves. Next $\int_{-L_1} \mathbf{v} \bullet \mathbf{T}\, ds = -\int_{L_1} \mathbf{v} \bullet \mathbf{T}\, ds$ and $\int_{-L_3} \mathbf{v} \bullet \mathbf{T}\, ds = -\int_{L_3} \mathbf{v} \bullet \mathbf{T}\, ds$. Finally we have used the additivity of paths in line integrals (property P4) to get $\int_{C_k} \mathbf{v} \bullet \mathbf{T}\, ds$ from the $\int_{L_i} \mathbf{v} \bullet \mathbf{T}\, ds$.

There is a class of vector fields for which the line integral only depends on the starting point and the ending point. These are precisely the conservative fields. To see this, let $\mathbf{v} = \nabla p$ so $\mathbf{v}$ is conservative and p is a potential function for it. Let $g(t) = p\big(\mathbf{r}(t)\big)$. Then $g'(t) = \nabla p\big(\mathbf{r}(t)\big) \bullet \mathbf{r}'(t)$ by the Chain Rule and so

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = \int_a^b \mathbf{v}\big(\mathbf{r}(t)\big) \bullet \mathbf{r}'(t)\, dt = \int_a^b \nabla p\big(\mathbf{r}(t)\big) \bullet \mathbf{r}'(t)\, dt = \int_a^b g'(t)\, dt = g(b) - g(a)$$

so

$$\int_C \mathbf{v} \bullet \mathbf{T}\, ds = p\big(\mathbf{r}(b)\big) - p\big(\mathbf{r}(a)\big)$$

This result is called the *Fundamental Theorem of Line Integrals* and has many uses, both practical and theoretical.

One theoretical use is to show that some fields are not conservative. For example, we just showed that $\langle y, 2x\rangle$ is not conservative since C_1 and C_2 have the same starting points and the same ending points but the two line integrals are not equal.

The converse is also true. Any vector field $\mathbf{v}$ such that the value of $\int_C \mathbf{v} \bullet \mathbf{T}\, ds$ only depends on the starting point and the ending point of C and not on the actual curve is conservative. Here we are require this for all curves which lie in the domain of $\mathbf{v}$. This seems a largely theoretical result since you have no practical way to check the hypothesis unless you actually see a potential function in which case you already know the field is conservative.

2. Conservative vector fields

Clairaut's Theorem supplies another way to check if a field is conservative which is much more practical. If $\langle M, N\rangle = \nabla p$ then $M = \dfrac{\partial p}{\partial x}$ and $N = \dfrac{\partial p}{\partial y}$. Note $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 M}{\partial x\, \partial y}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 N}{\partial y\, \partial x}$. If these are both continuous, Clairaut's Theorem says

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The continuity hypothesis is rarely a problem and so we only need to compute a couple of partials. For the field in the last section, $\langle y, 2x\rangle$, $\dfrac{\partial M}{\partial y} = \dfrac{\partial y}{\partial y} = 1$ whereas $\dfrac{\partial N}{\partial x} = \dfrac{\partial 2x}{\partial x} = 2$ so here is another reason why $\langle y, 2x\rangle$ is not conservative.

You can verify the result for fields in space.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

provided the displayed functions are continuous.

It would be really nice if this necessary condition were sufficient but alas it is not.

Example. Let $\mathbf{v} = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$.

$M = -\dfrac{y}{x^2+y^2}$, $\dfrac{\partial M}{\partial y} = -\dfrac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$N = \dfrac{x}{x^2+y^2}$, $\dfrac{\partial N}{\partial x} = \dfrac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

so

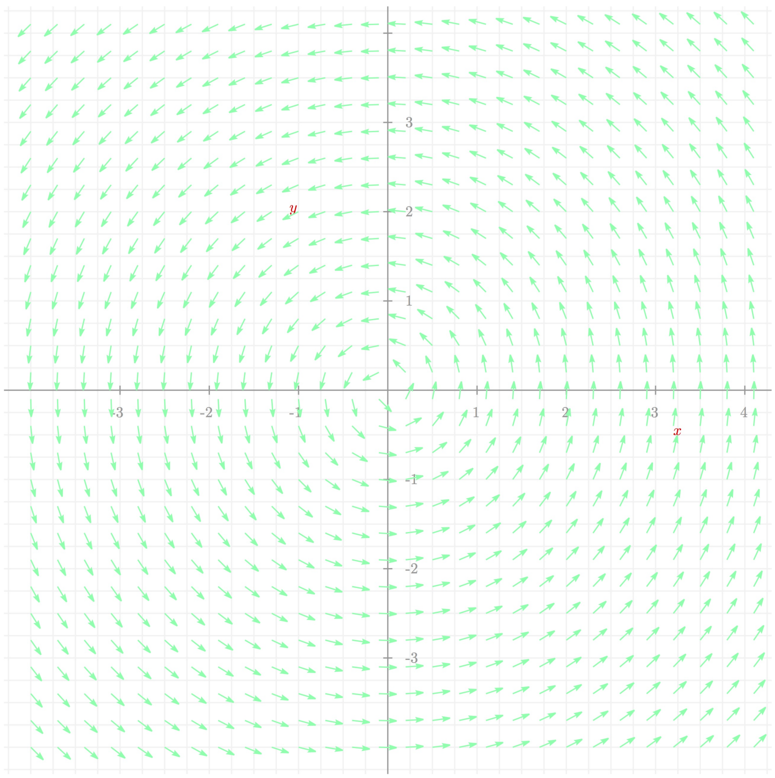
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Let us do $\int_C \mathbf{v} \bullet \mathbf{T}\, ds$ for this $\mathbf{v}$ around the unit circle centered at the origin. One parametrization for C is $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ $0 \leqslant t \leqslant 2\pi$ $d\mathbf{r} = \langle -\sin t, \cos t\rangle\, dt$ and $\mathbf{v}\big(\mathbf{r}(t)\big) = \langle -\sin t, \cos t\rangle$ so

$$\int_C \mathbf{v} \bullet d\mathbf{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \bullet \langle -\sin t, \cos t\rangle\, dt = 2\pi$$

Hence $\left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ is not conservative.

Here is a picture of the associated *direction field* (the vector field rescaled so that all vectors having length 1).



$$\mathbf{v} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

Can you see from the figure why $\int_C \mathbf{v} \bullet d\mathbf{r} \neq 0$? Click here for the answer.

Fields which satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ or the 3 dimensional version above are called *closed* and Clairaut's Theorem says that conservative fields are closed provided the mixed partials of a potential function are continuous. The problem of which closed fields are conservative started modern algebraic topology.

One sure way to show that a field is conservative is to exhibit a potential function for it. Let us start with the 2 dimensional case. Fix a field $\langle M, N\rangle$. We can start with either component so let us find a function $p_1(x, y)$ such that $\frac{\partial p_1(x,y)}{\partial x} = M$. This is a first year calculus problem so have at it. Once you have found one solution, $p_1(x, y)$ all solutions can be found by adding a constant (in x) or equivalently, a function of y, so the potential function $p(x,y) = p_1(x,y) + g(y)$ for some function $g(y)$. Then $\dfrac{\partial p}{\partial y} = N$ or $\dfrac{\partial p_1(x,y)}{\partial y} + g'(y) = N$ so $g'(y) = N - \dfrac{\partial p_1(x,y)}{\partial y}$.

This equation has a solution provided $h(x,y) = N - \dfrac{\partial p_1(x,y)}{\partial y}$ is a continuous function of y only and this is guaranteed if $\dfrac{\partial h(x,y)}{\partial x} = 0$. But with our mixed partial assumptions

$$\frac{\partial h(x,y)}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial^2 p_1}{\partial x\,\partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 p_1}{\partial y\,\partial x} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

If M and N are differentiable functions and if the field $\langle M, N\rangle$ is closed, then there is a potential function and you can find it if you can do some first year calculus integrals.

There is a similar result in three variables. The calculation takes one more step.

Example. Let $\langle y, x\rangle$. Check (easily) that$\langle y, x\rangle$ is closed. To find p, first solve $\dfrac{\partial p}{\partial x} = y$ or $p = xy + g(y)$. Then solve $\dfrac{\partial\, xy + g(y)}{\partial y} = x$ or $g'(y) = 0$ so $p = xy$ is one potential function and any other looks like $xy + c$.

Actually, there is no reason to check if the field is closed. Here's an example of what happens if it is not.

Example. Let $\langle y, 2x\rangle$. To find p, first solve $\dfrac{\partial p}{\partial x} = y$ or $p = xy + g(y)$. Then solve $\dfrac{\partial\, xy + g(y)}{\partial y} = 2x$ or $x + g'(y) = 2x$ so $g'(y) = x$ and this is impossible. This sort of issue is your clue that the original field is not closed.

Here is another example

Example. Let $\langle x^2 + y^2, 2xy\rangle$. This time let's start with $\dfrac{\partial p}{\partial y} = 2xy$ so $p = xy^2 + h(x)$ and then $\dfrac{\partial\, xy^2 + h(x)}{\partial x} = y^2 + h'(x) = x^2 + y^2$ so $h'(x) = x^2$ and $p = xy^2 + \dfrac{x^3}{3} + c$.

Here is a 3 dimensional example

Example. Let $\left\langle \dfrac{2xy}{z} + z, \dfrac{x^2}{z}, -\dfrac{x^2 y}{z^2} + x + e^z \right\rangle$. Start with $\dfrac{\partial p}{\partial y} = \dfrac{x^2}{z}$ so $p = \dfrac{x^2 y}{z} + h(x,z)$. Then solve $\dfrac{\partial\, \frac{x^2 y}{z} + h(x,z)}{\partial x} = \dfrac{2xy}{z} + \dfrac{\partial h}{\partial x} = \dfrac{2xy}{z} + z$ so $\dfrac{\partial h}{\partial x} = z$ and $h(x,z) = xz + g(z)$ and finally $\dfrac{\partial\, \frac{x^2 y}{z} + xz + g(z)}{\partial z} = -\dfrac{x^2 y}{z^2} + x + g'(z) = -\dfrac{x^2 y}{z^2} + x + e^z$ and $g'(z) = e^z$ so $p = \dfrac{x^2 y}{z} + xz + e^z + c$.

Let us try to apply this technique to the closed form $\mathbf{v} = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$.

$\dfrac{\partial p}{\partial y} = \dfrac{x}{x^2+y^2}$ so $p = \arctan\left(\dfrac{y}{x}\right) + g(x)$ and

$$\frac{\partial p}{\partial x} = \frac{\partial \arctan\left(\frac{y}{x}\right) + g(x)}{\partial x} = \frac{-\frac{y}{x^2}}{1 + \left(\frac{y}{x}\right)^2} + g'(x) = -\frac{y}{x^2+y^2}$$

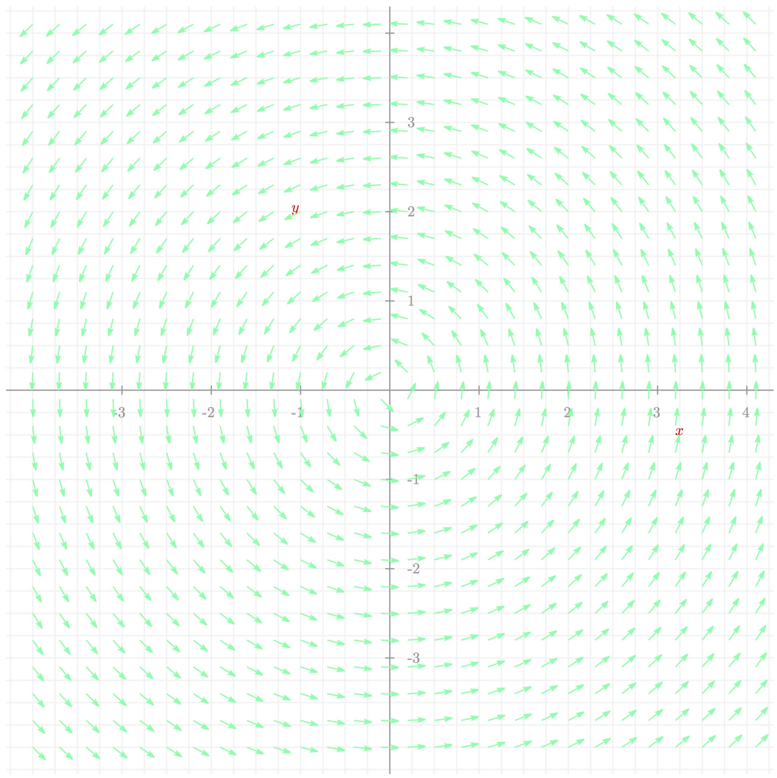
so

$$p(x,y) = \arctan\left(\frac{y}{x}\right)$$

is a potential function. But $\mathbf{v}$ has no potential function. What is wrong? Click here for the answer.

Appendix A. Line integral question.

Recall the question.



$$\mathbf{v} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

is a direction-field and we asked why is $\displaystyle\int_C \mathbf{v} \bullet d\mathbf{r} \neq 0$ where C is the unit circle centered at the origin.

The answer is that C is a flow line for the direction field and hence a flow line for the field. Clearly $\mathbf{v} \bullet \mathbf{T} > 0$ along the curve and hence $\displaystyle\int_C \mathbf{v} \bullet d\mathbf{r} > 0$ if we traverse the circle counterclockwise.

From the figure one suspects C is a flow line and clearly it is close one. With no calculation it is pretty clear that $\mathbf{v} \bullet \mathbf{T} > 0$ along the curve and hence $\displaystyle\int_C \mathbf{v} \bullet d\mathbf{r} > 0$.

To see why C is a flow line, note $\nabla(x^2+y^2) = \langle 2x, 2y \rangle$ is a normal vector to C and so $\langle -y, x \rangle$ is a tangent vector. Along C, $x^2+y^2 = 1$ so $\langle -y, x \rangle = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ which shows C is a flow line for the field.

Appendix B. Why is our "potential" function not a potential function.

Recall the issue. $\mathbf{v} = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ is a closed vector field which is not conservative.

$$p = \arctan\left(\frac{y}{x}\right)$$

$$\nabla p = \left\langle \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}, \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} \right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \mathbf{v}$$

which sure looks like the definition of conservative.

The solution is to notice that p is not defined on the same domain as $\mathbf{v}$. The domain of $\mathbf{v}$ is $\mathbb{R}^2 - \{(0,0)\}$ but the domain of p is $R = \mathbb{R}^2 - \{(0,y) \big| -\infty < y < \infty\}$, i.e. $\mathbb{R}^2$ minus the y-axis. Equivalently $R = \{(x,y) \big| (x,y) \in \mathbb{R}^2 \; x \neq 0\}$

If we restrict $\mathbf{v}$ to R, then $\mathbf{v}$ is conservative. Note R is simply-connected and we have a theorem which says $\mathbf{v}$ restricted to R is conservative. We have actually exhibited a potential function for it.

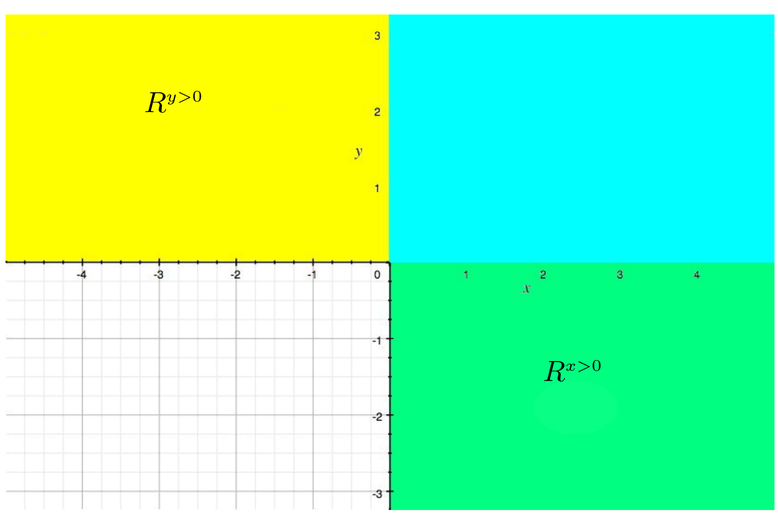
Recall $\arctan(\theta) + \arctan\left(\dfrac{1}{\theta}\right) = \dfrac{\pi}{2}$ for $\theta > 0$. Therefore $\arctan\left(\dfrac{y}{x}\right) = \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{y}\right)$ if $\dfrac{y}{x} > 0$. Hence $p_1 = -\arctan\left(\dfrac{x}{y}\right)$ is also a potential function for $\mathbf{v}$: i.e. $\nabla p_1 = \mathbf{v}$. Hence $\mathbf{v}$ is also conservative on the region $R_1 = \mathbb{R}^2 - \{(x,0) \big| -\infty < x < \infty\} = \{(x,y) \in \mathbb{R}^2 \big| y \neq 0\}$.

Let $S = \mathbb{R}^2 - \{(x,0) \;\Big|\; -\infty < x < 0\}$. Then $S = R^{y>0} \cup R^{x>0}$ where $R^{x>0} = \{(x,y) \in \mathbb{R}^2 \;\Big|\; x > 0\}$ and $R^{y>0} = \{(x,y) \in \mathbb{R}^2 \big| y > 0\}$. Note $R^{x>0} \cap R^{y>0} = \{(x,y) \in \mathbb{R}^2 \big| y > 0 \text{ and } x > 0\}$.

Define

$$q(x,y) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & (x,y) \in R^{>0} \\ \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{y}\right) & (x,y) \in R_1^{>0} \end{cases}$$

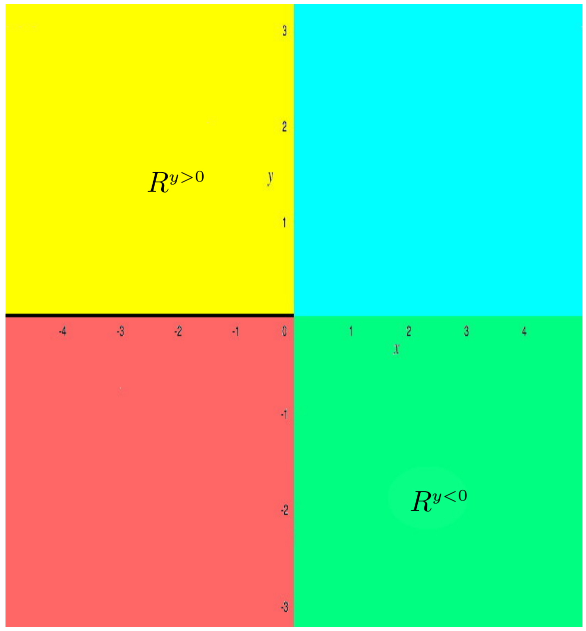
Then $\nabla q = \mathbf{v}$ on S so $\mathbf{v}$ is conservative on S which is simply-connected. The colored part of the plane below is S.



S

$R^{y>0}$ is the union of the yellow and blue regions, while $R^{x>0}$ is the union of the green and blue regions.

FInally, let $T = \{(x,y) \in \mathbb{R}^2 \mid -\infty < x < 0\}$.



T

The union of the red and green regions is $R^{y<0} = \{(x,y) \in \mathbb{R}^2 \mid y < 0\}$. If $\theta < 0$, $\arctan(\theta) + \arctan\left(\frac{1}{\theta}\right) = -\frac{\pi}{2}$ so $\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) = \pi + \arctan\left(\frac{y}{x}\right)$ in $R_1^{>0}$. Hence

$$q_1(x,y) = \begin{cases} \arctan\left(\frac{y}{x}\right) & (x,y) \quad y > 0 \text{ and } -\infty < x < \infty \\ \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) & (x,y) \quad x > 0 \text{ and } -\infty < y < \infty \\ \\ \pi + \arctan\left(\frac{y}{x}\right) & (x,y) \quad y < 0 \text{ and } -\infty < x < \infty \end{cases}$$

is a potential function for $\mathbf{v}$ on T. An equivalent way to define q_1 is

$$q_1(x,y) = \begin{cases} \arctan\left(\frac{y}{x}\right) & (x,y) \quad y > 0 \text{ and } -\infty < x < \infty \\ \frac{\pi}{2} & (x,0) \quad x > 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & (x,y) \quad y < 0 \text{ and } -\infty < x < \infty \end{cases}$$

It is harder to see that q_1 is a differentiable function using the second definition.

For any $x < 0$, $\lim_{t \to 0^-} q_1(x,t) = \pi$ and $\lim_{t \to 0^+} q_1(x,t) = 0$ so there is no way to further extend q_1 to include part of $x < 0$.