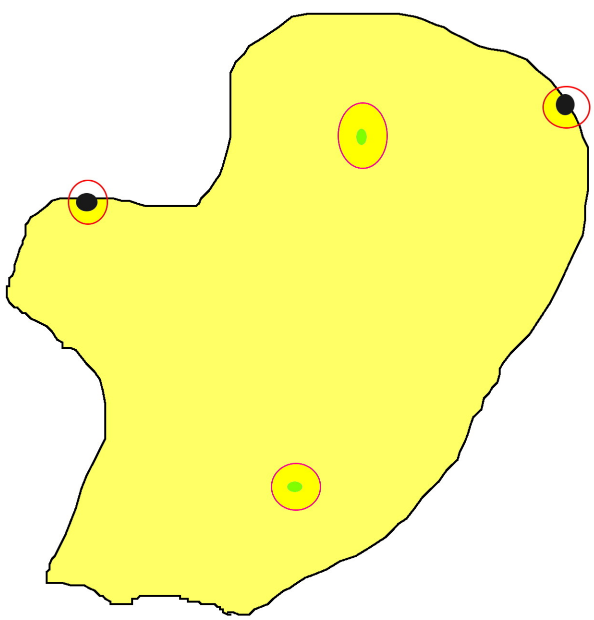
1. Definitions and main theorems

Max-min theory starts with a function f of a vector variable $\mathbf{x}$ and a subset D of the domain of f. So far when we have worked with functions we just use the entire domain of f, but often in max-min theory we want to restrict to some subset of the domain.

1.1. **Domains.** We need a bit of terminology related to the domain D. A point $\mathbf{a} \in D$ is an *interior point of* D provided there exists an $\epsilon > 0$ such that $\mathbf{b} \in D$ for all $\mathbf{b}$ satisfying $|\mathbf{b} - \mathbf{a}| < \epsilon$. Otherwise $\mathbf{a}$ is a *boundary point of* D.



The yellow part of the plane is the region D. The green dots are interior points because we have drawn a small circle around each one which is entirely contained in D, the yellow part. The black points are boundary points because no matter how small a circle we try to draw around either of them, the circle will contain points not in D. The circles around the interior points *proves* that they are interior points. The circles around the black points *suggests* that they are boundary points but does not prove it: maybe there is a smaller circle we could draw that is entirely in the yellow region. From the picture we can see this does not happen so they are boundary points.

You can also proceed as follows. A point $\mathbf{a} \in D$ is an interior point if and only if there exists an $\epsilon > 0$ such that the curve $\mathbf{u}t + \mathbf{a} \in D$ for all t with $|t| < \epsilon$ for all unit vectors u.

1.2. **Extrema.** Given a function of a vector variable $f(\mathbf{x})$, we say that a point $\mathbf{a} \in D$ is a *local maxima for f on D* provided there exists an $\epsilon > 0$ such that for every $\mathbf{b}$ with $|\mathbf{b} - \mathbf{a}| < \epsilon$ and $\mathbf{b} \in D$, $f(\mathbf{b}) \leqslant f(\mathbf{a})$.

> *Informally we say that $\boldsymbol{a}$ is a local maxima for f provided for all $\boldsymbol{b}$ near $\boldsymbol{a}$, $f(\boldsymbol{b}) \leqslant f(\boldsymbol{a})$.*

We say that a point $\mathbf{a} \in D$ is a *local minima for f on D* provided there exists an $\epsilon > 0$ such that for every $\mathbf{b}$ with $|\mathbf{b} - \mathbf{a}| < \epsilon$ and $\mathbf{b}$ in the domain of f, $f(\mathbf{b}) \geqslant f(\mathbf{a})$. We also say that in this case, f has a local minima at $\mathbf{a}$.

A number M is a *maximum value* for a function on a subset D of its domain provided $f(\mathbf{x}) \leqslant M$ for all $\mathbf{x} \in D$.

A number m is a *minimum value* for a function on a subset D of its domain provided $f(\mathbf{x}) \geqslant m$ for all $\mathbf{x} \in D$.

An *absolute maxima* on a subset D of the domain of f is a local maxima for f on D $\mathbf{a}$ such that $f(\mathbf{a})$ is a maximum for f on D.

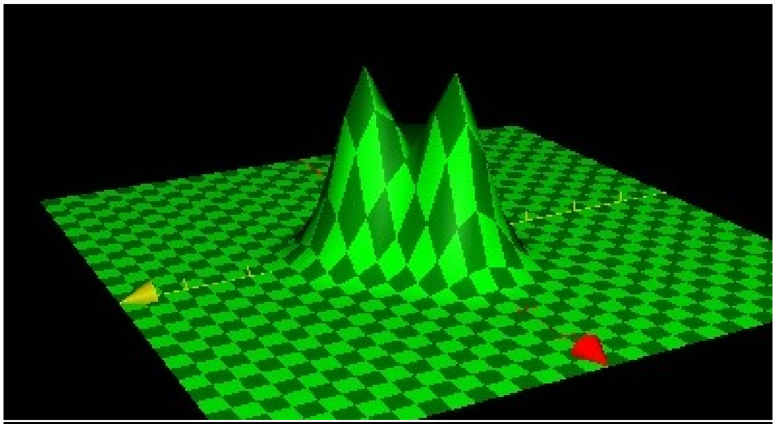
An *absolute minima* on a subset D of the domain of f is a local minima for f on D $\mathbf{a}$ such that $f(\mathbf{a})$ is a minimum for f on D.

This may seem like a lot of definitions, but a couple of observations may help. Maxima/minima are points (aka vectors) so that the value of f at the point has either a maximum or minimum value. The adjective *local* means that the property holds for all nearby points (in D): *absolute* means that the property holds for all points (in D).

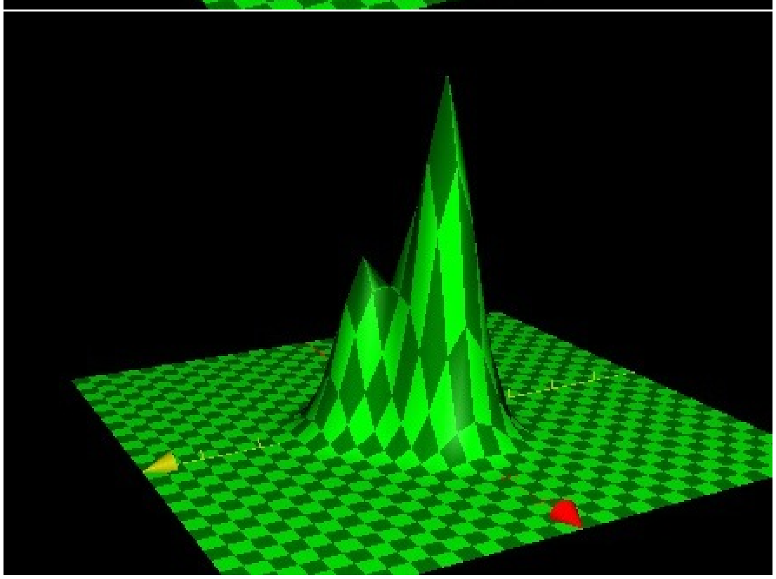
Because our techniques for locating maxima and minima are the same, it will be convenient to say that a point is a *local extrema* provided it is either a local maxima or a local minima.

Note that any absolute extrema is automatically a local extrema but not conversely. Hence if $\mathbf{a}$ is an absolute extrema we will rarely mention that it is also a local extrema.

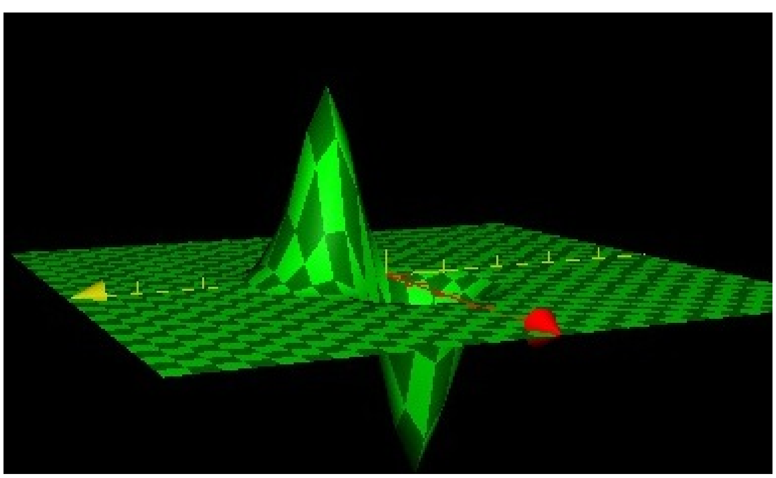
In the graphs below, the z-axis is missing and the red axis is the y-axis.



$z = 3(10^{-x^2+(y-1)^2}+10^{-(x-1)^2+y^2})$
Two maxima, both absolute. Maximum value is the value of the function at either maxima.



$z = 3(2{\cdot}10^{-x^2+(y-1)^2}+10^{-(x-1)^2+y^2})$
Two maxima, one absolute, one local. Maximum value is the value of the function at the taller of the the local maxima.



$z = 3(-10^{-x^2+(y-1)^2}+10^{-(x-1)^2+y^2})$
Two extrema, both absolute. One maxima, one minima. Maximum value is the value of the function at the maxima: Minimum value is the value of the function at the minima.

2. Locating local extrema

2.1. The one variable case. Recall that a differentiable function of one variable, $g(t)$, on an open interval (a, b) can have a local extrema only at a point where $g'(t) = 0$, a so called critical point.

The converse is not true. The function $g(t) = t^2$ has a minima at $t = 0$, the function $g(t) = -t^2$ has a maxima at $t = 0$ and the function $g(t) = t^3$ has a critical point at $t = 0$ which is neither a maxima nor a minima.

The First Derivative Test for a critical point to be an extrema is the following. If a is a critical point (ie. $g'(a) = 0$) and if, for all small positive t,

- $g'(a - t) > 0$ and $g'(a + t) < 0$, then a is a local maxima.
- $g'(a - t) < 0$ and $g'(a + t) > 0$, then a is a local minima.
- Otherwise a is a critical point which is not a local extrema.

The Second Derivative Test for a critical point to be an extrema is the following. If a is a critical point (ie. $g'(a) = 0$) and if

- $g''(a) < 0$ then a is a local maxima.
- $g''(a) > 0$, then a is a local minima.
- $g''(a) = 0$ then a can be either a local maxima, a local minima, or not be a local extrema at all.

2.2. The multi-variable case.

Suppose that $\boldsymbol{a}$ *is an interior point of* D *and that* f *is differentiable on* D. *If* $\boldsymbol{a}$ *is a local extrema, then* $\nabla f(\boldsymbol{a}) = \boldsymbol{0}$.

Points in the interior of D where $\nabla f(\mathbf{a}) = \mathbf{0}$ are called *critical points* and the general strategy in the multi-variable case is the same as the one variable case.

(1) Find the critical points.
(2) Determine if a critical point is a local maxima, a local minima, or neither.

2.3. Examples of finding critical points.

Example. Find the critical point(s) of $f(x, y) = x^3 + y^3 - 3xy$.

$\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x\rangle$ so

(1) $3x^2 - 3y = 0$
(2) $3y^2 - 3x = 0$

Solve one equation for one of the variables, say using (1), $y = x^2$. Plug into (2) to get $x^4 = x$. Then $x = 0$, or 1 and the corresponding points are $(0, 0)$, $(1, 1)$.

Example. $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2$.

$\nabla f = \langle 6xy - 12x, 3y^2 + 3x^2 - 12y\rangle$

$$6xy - 12x = 0 \tag{1}$$

$$3y^2 + 3x^2 - 12y = 0 \tag{2}$$

From (1), $x(y-2) = 0$ so $x = 0$ or $y = 2$.

If $x = 0$, (2) becomes $3y^2 - 12y = 0$ or $y(y-4) = 0$ so points are $(0,0)$ and $(0,4)$.

If $y = 2$, $12 + 3x^2 - 24 = 0$ or $3x^2 = 12$, $x^2 = 4$ or $x = \pm 2$ so critical points are $(2,2)$ and $(-2,2)$.

3. What kind of extrema is a particular critical point?

Some theory is needed to identify the type of the critical point. Here is a two-variable version of the Second Derivative Test.

Given any point, define the *determinant of the Hessian of f*, at least in the two variable case. as follows.

$$\mathcal{H}(f) = \det \begin{vmatrix} \dfrac{\partial^2 f}{\partial^2 x} & \dfrac{\partial^2 f}{\partial y\,\partial x} \\ \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial^2 y} \end{vmatrix} = \frac{\partial^2 f}{\partial^2 x}\frac{\partial^2 f}{\partial^2 y} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2$$

At a critical point, $\mathbf{a}$ we have the Second Derivative Test. Suppose f has continuous second partials.

(1) If $\mathcal{H}(f)(\mathbf{a}) > 0$ then $\mathbf{a}$ is a local extrema.
   (a) If $\dfrac{\partial^2 f}{\partial^2 x}(\mathbf{a}) > 0$, $\mathbf{a}$ is a local minima.
   (b) If $\dfrac{\partial^2 f}{\partial^2 x}(\mathbf{a}) < 0$, $\mathbf{a}$ is a local maxima.

(2) If $\mathcal{H}(f)(\mathbf{a}) < 0$ then $\mathbf{a}$ is called a *saddle point.*

(3) If $\mathcal{H}(f)(\mathbf{a}) = 0$ then the type of $\mathbf{a}$ is undetermined.

Notice that if $\mathcal{H}(f)(\mathbf{a}) > 0$, $\dfrac{\partial^2 f}{\partial^2 x}$ and $\dfrac{\partial^2 f}{\partial^2 y}$ have the same sign and neither is 0, so we could have used $\dfrac{\partial^2 f}{\partial^2 y}$ in (1).

Put another way

(1): det > 0 local extrema, sign of the diagonal entries tells which sort.
(2): det < 0 not a local extrema but a saddle point.
(3): det $= 0$ no idea.

Example. $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2$.

$\nabla f = \langle 6xy - 12x, 3y^2 + 3x^2 - 12y \rangle$

$$6xy - 12x = 0 \tag{1}$$

$$3y^2 + 3x^2 - 12y = 0 \tag{2}$$

$$\mathcal{H}(f) = \det \begin{vmatrix} 6y - 12 & 6x \\ 6x & 6y - 12 \end{vmatrix}$$

Critical points are $(0,0)$, $(0,4)$, $(2,2)$ and $(-2,2)$.

$(0,0)$: $\mathcal{H}(f)(0,0) = \det \begin{vmatrix} -12 & 0 \\ 0 & -12 \end{vmatrix} = 144 > 0$ and $-12 < 0$, so local maxima.

$(0,4)$: $\mathcal{H}(f)(0,4) = \det \begin{vmatrix} 12 & 0 \\ 0 & 12 \end{vmatrix} = 144 > 0$ and $12 > 0$, so local minima.

$(2,2)$: $\mathcal{H}(f)(2,2) = \det \begin{vmatrix} 0 & 12 \\ 12 & 0 \end{vmatrix} = -144 < 0$, so saddle point.

$(-2,2)$: $\mathcal{H}(f)(-2,2) = \det \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$, so saddle point.

4. No easy First Derivative Test

Consider the function $f(x,y) = (y-4x^2)(y-x^2)$. This function is a polynomial and so is as nice as it gets in terms of differentiability, continuity, etc..

$\dfrac{\partial f}{\partial x} = (-8x)(y-x^2) + (y-4x^2)(-2x) = -10xy + 16x^3$

$\dfrac{\partial f}{\partial y} = 1(y-x^2) + (y-4x^2)1 = 2y - 5x^2$

Critical point(s):

$0 = -10xy + 16x^3$ AND $0 = 2y - 5x^2$.

Clearly $(0,0)$ is a critical point and it is the only critical point with $x = 0$.

If $x \neq 0$, then $0 = -10y + 16x^2$ and still $0 = 2y - 5x^2$ so eliminating y, $0 = -9x^2$ so $x = 0$ which is a contradiction and so there are no solutions with $x \neq 0$.

Hessian:

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -10y + 48x^2 & -10x \\ -10x & 2 \end{pmatrix}$$

At the critical point, the Hessian is

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

which has determinant 0.

Consider what happens as the critical point is approached along the line $\mathbf{r}(t) = \langle a,b\rangle t$ where $|\langle a,b\rangle| > 0$. Let $g(t) = f\big(\mathbf{r}(t)\big) = f(at,bt) = (bt - 4a^2t^2)(bt - at^2) = t^2(b-4a^2t)(b-a^2t) = t^2(b^2 - 5ba^2t + 4a^4t^2)$.

Check

$$g'(t) = 2b^2t - 15a^2bt^2 + 16a^4t^3$$

so g has a critical point at $t = 0$.

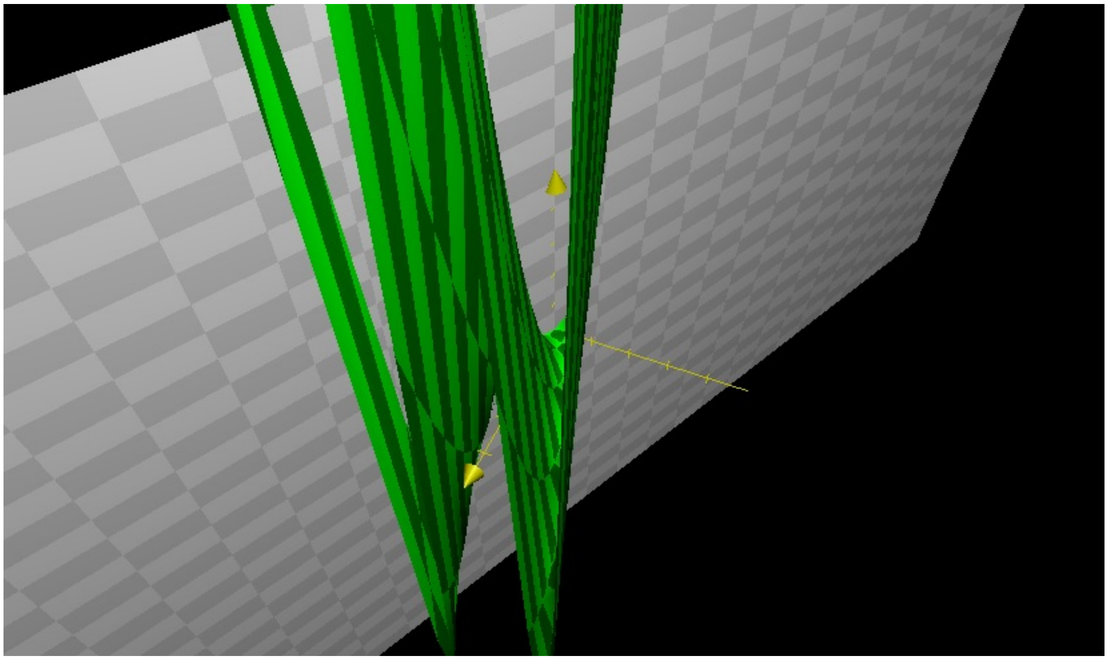
Check

$$g''(t) = 2b^2 - 30a^2bt + 48a^4t^2$$

If $b = 0$ then $a \neq 0$ and $g(t) = 4a^4t^4$ has a minimum at $t = 0$. If $b \neq 0$, $g''(0) > 0$ so $g(t)$ also has a minimum at $t = 0$.

Hence along any line through the critical point, the function has a minimum.

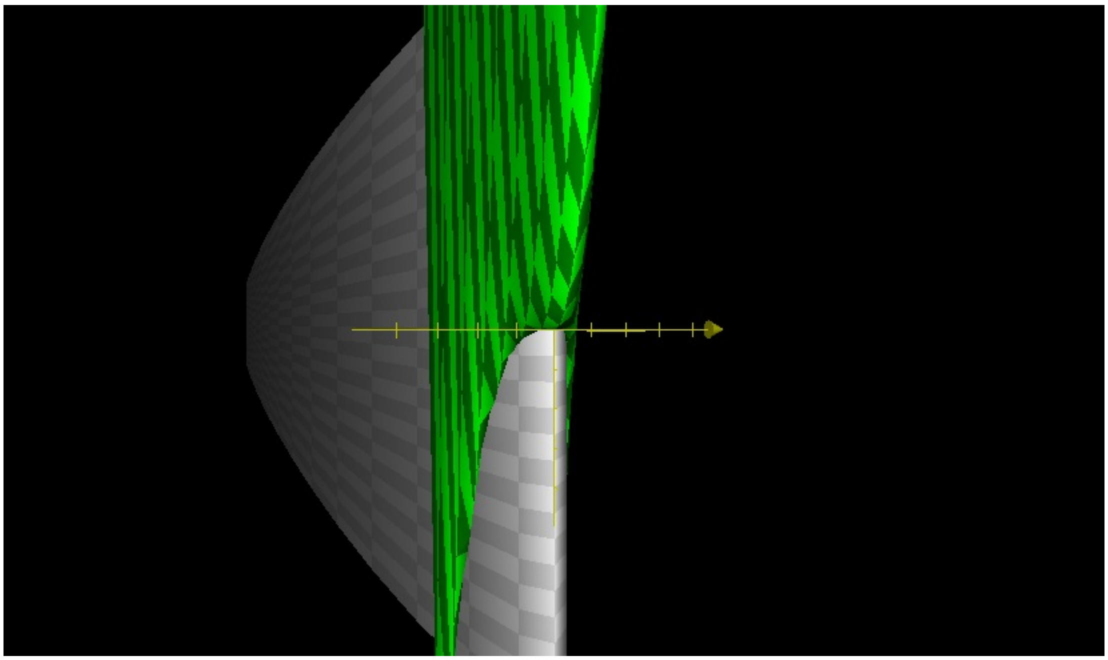
Here is a view of the picture when $a = 1$, $b = 2$.



The green shaded surface is the graph of $f(x, y) = (y - 4x^2)(y - x^2)$. The gray shaded plane is $\langle t, 2t, z\rangle$ and the intersection is the curve $g(t)$ for $a = 1$, $b = 2$.

Now consider the family of parabolas, $\mathbf{r}(t) = \langle t, bt^2 \rangle$. This time $g(t) = f\big(\mathbf{r}(t)\big) = f(t, bt^2) = (bt^2 - 4t^2)(bt^2 - t^2) = t^4(b-4)(b-1)$. Along the parabola $b = 1$ or $b = 4$, the paths look level as you approach the origin. For $b > 4$ or $b < 1$, the path still has the critical point as a minimum.

BUT for $1 < b < 4$, the path to the critical point has a maximum!



This time the gray surface is the parabolic-cylinder $\langle t, 2t^2, z \rangle$ and the intersection is the curve $g(t) = -2t^4$.

You can sometimes draw conclusions from looking at lines through the critical point.

> *If you can find one direction for which the curve through the critical point in that direction has a local maxima at the critical point and another direction for which the curve through the critical point in that direction has a local minima at the critical point, then for sure the critical point is a saddle point.*

5. More weird examples

Example. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2$ is an example of a polynomial which has two local maxima but no local minima. For functions of one variable this is impossible as long as the derivative is continuous.

$\nabla f = \langle -2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1), -2(x^2y-x-1)(x^2)\rangle$

$$-2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1) = 0 \tag{1}$$
$$-2(x^2y-x-1)x^2 = 0 \tag{2}$$

Hence $x=0$ or $x^2y - x - 1 = 0$. If $x = 0$, (1) has no solution. Hence $x \neq 0$ and $y = \frac{x+1}{x^2}$. Then (1) becomes

$$-2(x^2-1)(2x) - 2(x^2\frac{x+1}{x^2} - x - 1)(2x\frac{x+1}{x^2} - 1) = 0$$
$$-2(x^2-1)(2x) = 0$$

Since $x \neq 0$, $x = \pm 1$ and there are two critical points

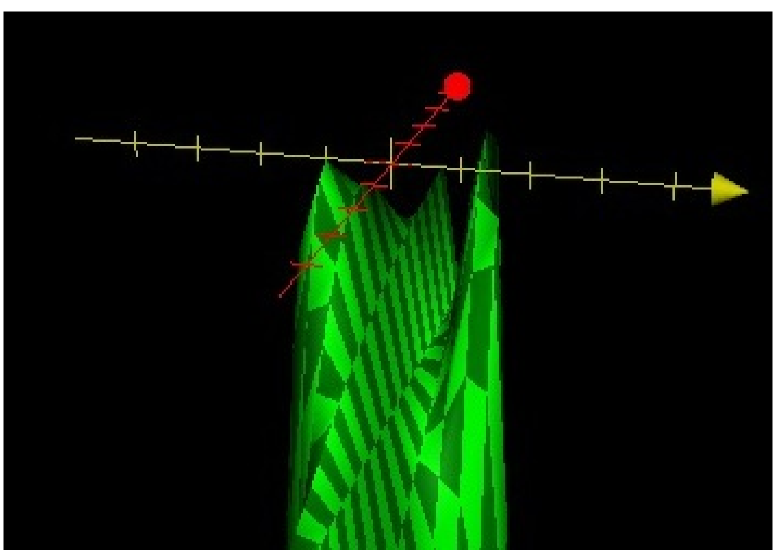
$$(1,2) \text{ and } (-1,0)$$

I will leave it to you to calculate that the Hessian is

$$\mathcal{H}(f) = \det\begin{vmatrix} -12x^2 - 12x^2y^2 + 12xy + 4y + 2 & -8x^3y + 6x^2 + 4x \\ -8x^3y + 6x^2 + 4x & -2x^4 \end{vmatrix}$$

$(1,2)$: $\mathcal{H}(f)(1,2) = \det\begin{vmatrix} -26 & -6 \\ -6 & -2 \end{vmatrix} = 26 > 0$ and $-26 < 0$ so local maxima.

$(-1,0)$: $\mathcal{H}(f)(-1,0) = \det\begin{vmatrix} -10 & 2 \\ 2 & -2 \end{vmatrix} = 16 > 0$ and $-10 < 0$ so local maxima.



Example. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is an example of a function which has one local maxima, no absolute maxima and no local minima. Again, this can not happen in one variable as long as the derivative is continuous.

$\nabla f = \langle 3e^y - 3x^2, 3xe^y - 3e^{3y} \rangle$

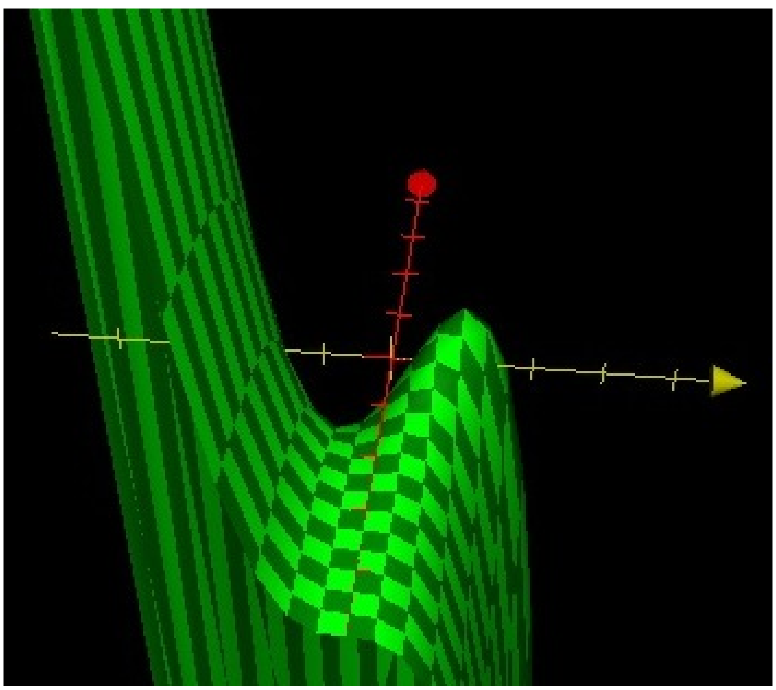
$$3e^y - 3x^2 = 0 \tag{1}$$

$$3xe^y - 3e^{3y} = 0 \tag{2}$$

Since $e^y \neq 0$, (2) shows $x = e^{2y}$ and (1) becomes $e^y = e^{4y}$. Hence $y = 0$ and $(1,0)$ is the only critical point.

$$\mathcal{H}(f) = \det \begin{vmatrix} -6x & 3e^y \\ 3e^y & 3xe^y - 9e^{3y} \end{vmatrix}$$

so $\mathcal{H}(f)(1,0) = \det \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 > 0$ and $-6 < 0$ so local maxima.



Here is a tip for minimizing your work and maximizing your accuracy with the Second Derivative Test. When you compute the determinant of the Hessian,

$$\mathcal{H}(f) = \det \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y\,\partial x} \\ \dfrac{\partial^2 f}{\partial y\,\partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

DO NOT multiply it out! It is relatively easy to plug the critical point values into each piece of the matrix and only then should you take the determinant. It is much easier to take the determinant of a matrix of four numbers than a matrix of four functions and, if the determinant happens to be positive you will see the diagonal entries which are what you need to decide if you have a local maxima or a local minima.

Try multiplying out the determinant of the Hessian in the example on page 10 before you plug in if you want to add five or ten minutes to the time needed to find the types of the critical points.

6. Absolute extrema on closed regions

There is a theorem from one variable calculus which says that a continuous function on a closed interval has a maxima and a minima. The extrema occur either at a local extrema in the interior of the interval or at the endpoints of the interval. Something similar is true in more than one variable.

A region D is *closed* if given any curve $\mathbf{r}(t)\colon (0,1) \to D$ such that $\lim_{t\to 0^+} \mathbf{r}(t)$ exits, then $\lim_{t\to 0^+} \mathbf{r}(t) \in D$. In practice it is usually easy to see if a region in the plane is closed or not.

A region D is *compact* or *bounded* if it is contained in a disk of some radius.

Any continuous function on a closed, bounded region has an absolute maxima and an absolute minima.

As a matter of logic it follows that the extrema are either at interior points or along the boundary of D. This means there is a different procedure for locating absolute extrema.

To locate the absolute extrema of a function f on a closed bounded region D.

Make a list of points where an absolute extrema might occur. It is not important if you have too many points on your list but it is vital that you do not miss any where an extrema might occur. The technique below almost always produces a list with points on it that are not absolute extrema.

(1) Locate the critical points of f in the interior. You do not need to do the Second Derivative Test at these points when doing this problem. Add these points to your list.
(2) Locate points on the boundary of D which are potential points where an absolute extrema of f might occur. We will discuss one way to do this today and introduce another method next time.
(3) Evaluate the function at each of the points on your list and see which value is biggest and which is smallest.

One way to do (2) is to parametrize the boundary by $\boldsymbol{r}(t)$, $a \leqslant t \leqslant b$. *The value of* f *along the boundary is* $g(t) = f\big(\boldsymbol{r}(t)\big)$ *which is a one variable problem. Solve* $g'(t) = 0$. *If* t_0 *is a critical point in* (a, b) *then* $\boldsymbol{r}(t_0)$ *goes on the list as do* $\boldsymbol{r}(a)$ *and* $\boldsymbol{r}(b)$.

Example. Consider again the function $f(x, y) = x^3 + y^3 - 3xy$ and find the absolute extrema of f on the region D which is the inside of the triangle with vertices $(0, 3)$, $(3, 3)$ and $(0, 0)$.

(1) We found the critical points of f above $(0, 0)$ and $(1, 1)$. Neither of these points is in the interior of D

(2) In chapter 13 we found out how to parametrize a line segment. Hence the boundary of D, which is the triangle, can be parametrized in 3 pieces

(a) $\mathbf{r}_1(t) = t\,\langle 0, 0\rangle + (1 - t)\,\langle 0, 3\rangle = \langle 0, 3 - 3t\rangle$ $0 \leqslant t \leqslant 1$.
(b) $\mathbf{r}_2(t) = t\,\langle 0, 3\rangle + (1 - t)\,\langle 3, 3\rangle = \langle 3 - 3t, 3\rangle$ $0 \leqslant t \leqslant 1$.
(c) $\mathbf{r}_3(t) = t\,\langle 3, 3\rangle + (1 - t)\,\langle 0, 0\rangle = \langle 3t, 3t\rangle$ $0 \leqslant t \leqslant 1$.

(a): $g_1(t) = f\big(\mathbf{r}_1(t)\big) = 0^3 + (3 - 3t)^3 - 3 \cdot 0 \cdot (3 - 3t) = 27(1 - t)^3$. Critical points are $g_1'(t) = 0$ or $81(1 - t)^2(-1) = 0$ or $t = 1$. This is an endpoint and the general theory says we need to consider both endpoints, $t = 0$, 1 or $(0, 0)$ and $(0, 3)$.

(b): $g_2(t) = f\big(\mathbf{r}_2(t)\big) = 27(1 - t)^3 + 27 - 27 \cdot (1 - t)$. Critical points are $g_2'(t) = 0$ or $81(1-t)^2(-1) + 27 = 0$ or $(1-t)^2 = 1/3$ or $t = 1 \pm \frac{1}{\sqrt{3}}$.

We are looking for solutions between 0 and 1 and only $t = 1 - \frac{1}{\sqrt{3}}$ is in this range. The corresponding point is $(\sqrt{3}, 3)$. The general theory says we still need to consider both endpoints, $t = 0$, 1 or $(0, 3)$ and $(3, 3)$.

(c): $g_3(t) = f\big(\mathbf{r}_3(t)\big) = (3t)^3 + (3t)^3 - 27t^2 = 27(2t^3 - t^2)$. Critical points are $g_3'(t) = 0$ or $27(6t^2 - 2t) = 0$ or $54t(3t - 1) = 0$ so $t = 0$ and $t = 1/3$. When $t = 0$ we are at an endpoint. When $t = 1/3$ we are at $(1, 1)$ The general theory says to include both endpoints endpoints, $(0, 0)$ and $(3, 3)$. Indeed, if a critical point of f is on the boundary, it will be a critical point of the parametrized boundary function.

(3)

point	$(0,0)$	$(1,1)$	$(3,3)$	$(\sqrt{3},3)$	$(0,3)$
value	0	-1	27	$27-6\sqrt{3}$	27

Hence the minimum value is -1 and the corresponding minima is at $(1, 1)$. The maximum value is 27 and the corresponding maxima is at $(0, 3)$ and $(3, 3)$.

Example. Find the max/min values of $f(x, y) = xy^2$ on the disk $x^2 + y^2 = 3$ in the first quadrant.

Critical Points: $\nabla f = \langle y^2, 2xy\rangle$ so there are infinitely many critical points, all of the form $(x, 0)$. These are boundary points for D so there are no critical points in the interior of D.

Along either axis, f is 0 so all points on both axis are on the list from the origin out to $\sqrt{3}$. The remaining part of the boundary is the circle of radius $\sqrt{3}$ in the first quadrant. There are two ways to parametrize this.

$\mathbf{r}_1(t) = \sqrt{3}\ \langle \cos t, \sin t\rangle$, $0 \leqslant t \leqslant \pi/2$ and
$\mathbf{r}_2(t) = \langle t, \sqrt{3 - t^2}\rangle$, $0 \leqslant t \leqslant \sqrt{3}$.
$g_1(t) = 3\sqrt{3} \cos t \sin^2 t = 3\sqrt{3}(\cos t - \cos^3 t)$
$g_2(t) = t(3 - t^2) = 3t - t^3$

$$g_1'(t) = 3\sqrt{3}(-\sin t+3\cos^2 t \sin t) = 3\sqrt{3}\big(-\sin t+3(1-\sin^2 t)\sin t\big) = 3\sqrt{3}\big(2\sin t-3\sin^3 t\big)$$

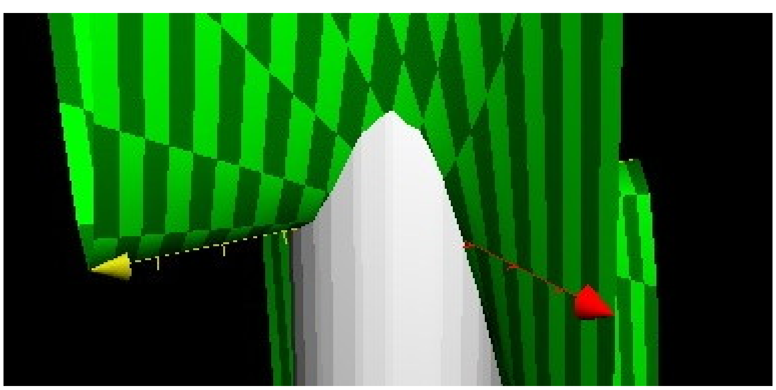
$$g_2'(t) = 3 - 3t^2$$

Critical points are at $\sin t = 0$ and $\sin t = \sqrt{\dfrac{2}{3}}$. These points are $(\sqrt{3}, 0)$ and $\sqrt{3}\left(\sqrt{\dfrac{1}{3}}, \sqrt{\dfrac{2}{3}}\right) = (1, \sqrt{2})$. $t = 0$ is one endpoint but we should include the other as well, $t = \pi/2$ and the point is $(0, \sqrt{3})$.

OR

Critical point is at $t = 1$ and the point is $(1, \sqrt{2})$ and the endpoints are $(\sqrt{3}, 0)$ and $(0, \sqrt{3})$.

Hence the minimum value is 0 and it occurs along both axes. The maximum value must occur at $(1, \sqrt{2})$ and the value is 2.



Example. Again consider the function $f(x, y) = x^3 + y^3 - 3xy$. Find the absolute extrema of f on the region D which is the disk centered at the origin of radius 5.

(1) The critical points of f $(0, 0)$ and $(1, 1)$ and both these points are in the interior of D.

(2) $\mathbf{r}(t) = 5\,\langle \cos(t), \sin(t)\rangle$, $0 \leqslant t \leqslant 2\pi$.

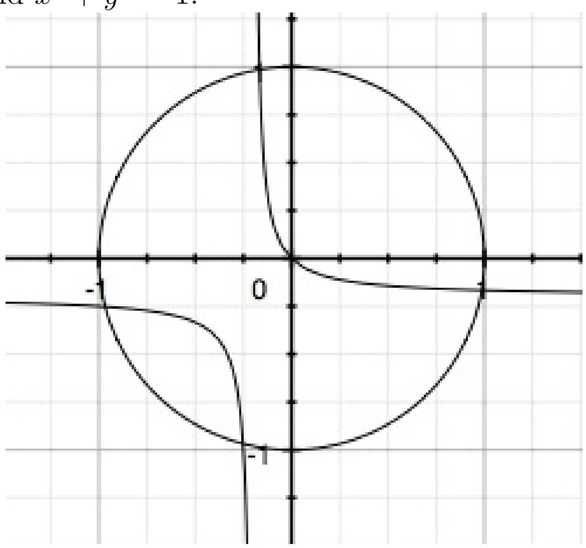
$$g(t) = 5^3 \cos^3 t + 5^3 \sin^3 t - 3 \cdot 5^2 \cos t \sin t$$

$$g'(t) = -3 \cdot 5^3 \cos^2 t \sin t + 3 \cdot 5^3 \sin^2 t \cos t + 3 \cdot 5^2 \sin^2 t - 3 \cdot 5^2 \cos^2 t =$$

$$3 \cdot 5^2 \big(5 \sin t \cos t(\sin t - \cos t) + \sin^2 t - \cos^2 t\big) =$$

$$3 \cdot 5^2 (\sin t - \cos t)(5 \sin t \cos t + \sin t + \cos t)$$

Hence the critical points are at $\pi/4$, $5\pi/4$ and the four angles you see in this picture of the intersection of two graphs, $5xy + x + y = 0$ and $x^2 + y^2 = 1$.



Label them θ_1, θ_2, θ_3, θ_4 going counterclockwise from $(1, 0)$.

You should also include the end points $t = 0$, 2π but since they both give the same point in the plane you only need one of them.

The list of points on the boundary is

(3) The values at the two interior critical points $(0, 0)$ and $(1, 1)$ are 0 and -1. By symmetry, if $(5, 0)$ were an absolute extrema, $(0, 5)$ would also be one and it is not on the list. Hence the maximum value must be greater than 5^3 and therefore it occurs at θ_1 and θ_4. Solve for θ_i to as many places as you need and plug into f.

$t =$	$\pi/4$	$5\pi/4$	0	θ_1	θ_2	θ_3	θ_4
point	$\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$	$\left(-\frac{5\sqrt{2}}{2}, -\frac{5\sqrt{2}}{2}\right)$	$(5, 0)$	(x_1, y_1)	(x_2, y_2)	(y_2, x_2)	(y_1, x_2)
value	$\frac{5^2}{2}\big(5\sqrt{2} - 3\big)$	$-\frac{5^2}{2}\big(5\sqrt{2} + 3\big)$	5^3	?	?	?	?

To find the minimum value is more difficult. Note $-\frac{5^2}{2}\big(5\sqrt{2}+3\big) < -5^3$ so it is hard to decide if the minimum value occurs at $5\pi/4$ or at θ_2 and θ_3, or perhaps at all three.