## 1. Parametric surfaces

A *parametrized surface* is, roughly speaking, a vector function of a vector variable

$$\mathbf{r}(u,w) = \langle x(u,w), y(u,w), z(u,w)\rangle : D \to \mathbb{R}^3$$

defined over some subset $D \subset \mathbb{R}^2$. The "roughly speaking" comes from the fact that we want to impose conditions on $\mathbf{r}$ so that the image is 2 dimensional and so really a surface. We will do this below but your informal ideas will suffice to get us through some examples.

**Example.** The plane through a point $\langle a,b,c\rangle$ containing two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

$$\mathbf{r}(u,w) = \langle a,b,c\rangle + u\mathbf{v}_1 + w\mathbf{v}_2$$

is one equation for that plane. It is also easy to describe the normal form of the plane. A normal vector is given by $\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2$ and so the equation is $\mathbf{N} \bullet \langle x,y,z\rangle = \mathbf{N} \bullet \langle a,b,c\rangle$. Review how to get a plane through 3 points, a plane containing a line and a point, ....

**Example.** The graph of $z = f(x,y)$ lying over $D$ in the $xy$ plane can be parametrized as

$$\mathbf{r}(u,w) = \langle u, w, f(u,w)\rangle$$

There are similar parametrizations for $y = g(x,z)$, $\mathbf{r}(u,w) = \langle u, g(u,w), w\rangle$ and $x = h(y,z)$, $\mathbf{r}(u,w) = \langle u, h(u,w), u, w\rangle$.

**Example.** The sphere of radius $a > 0$

$$\mathbf{r}(u,w) = \langle a\cos(u)\sin(w), a\sin(u)\sin(w), a\cos(w)\rangle \qquad 0 \leqslant u \leqslant 2\pi; \quad 0 \leqslant w \leqslant \pi$$

This is just spherical coordinates.

**Example.** If $\langle x(t), y(t)\rangle$ $a \leqslant t \leqslant b$ is a parametrized curve in the $xy$ plane then

$$\mathbf{r}(u,w) = \langle x(u), y(u), w\rangle \qquad a \leqslant u \leqslant b; \quad -\infty < w < \infty$$

is the cylinder along that curve.

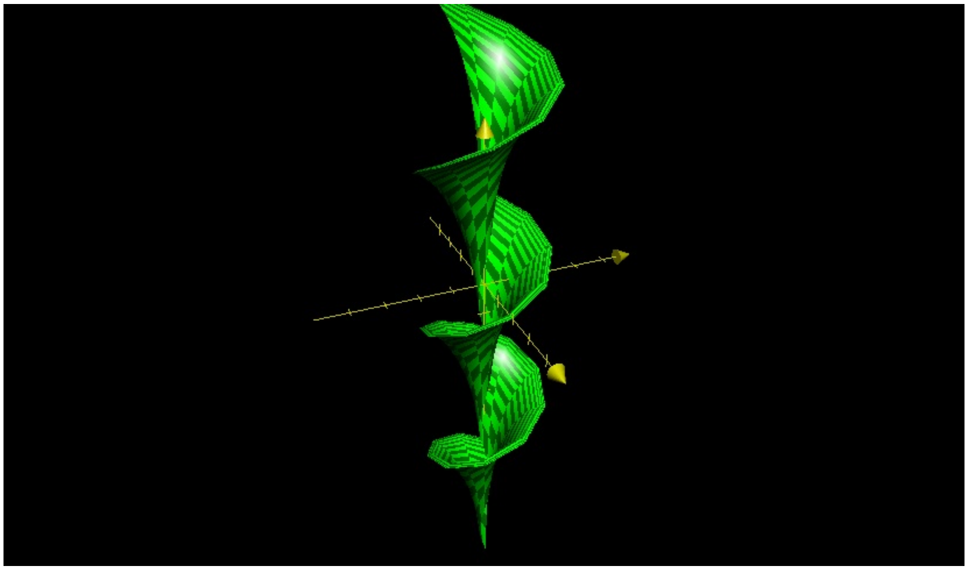
**Example.** If $\langle x(t), y(t)\rangle$ $a \leqslant t \leqslant b$ is a parametrized curve in the $xy$ plane then we can rotate about the $x$-axis to get a surface of revolution

$$\mathbf{r}(u,w) = \langle x(u), y(u)\cos(w), y(u)\sin(w)\rangle \qquad a \leqslant u \leqslant b; \quad 0 \leqslant w \leqslant 2\pi$$

We can also rotate about the $y$-axis to get a different surface of revolution

$$\mathbf{r}(u,w) = \langle x(u)\cos(w), y(u), x(u)\sin(w)\rangle \qquad a \leqslant u \leqslant b; \quad 0 \leqslant w \leqslant 2\pi$$

**Example.** Here is a picture of a surface.



A surface of Dini

$$\mathbf{r}(u,w) = \left\langle 2\cos(u)\sin(w), 2\sin(u)\sin(w), 2\left(\cos(w) + \ln\left(\tan\left(\frac{w}{2}\right)\right) + 0.4u\right)\right\rangle$$
$$-2\pi \leqslant u \leqslant 2\pi; \quad 0.1 < w < 2$$

The surface does not determine a parametrized surface any more than a line determines a parametric equation for it. Nevertheless, in order to do calculations we will need to have a parametrization.

**Example.** Consider the surface $z = x^2 + y^2 + 1$. It can be parametrized as

$$\mathbf{r}(u,w) = \left\langle u, w, u^2 + w^2 + 1\right\rangle \qquad -\infty < u < \infty; \quad -\infty < w < \infty$$

Using cylindrical coordinates it can also be parametrized as

$$\mathbf{r}(u,w) = \left\langle u\cos(w), u\sin(w), u^2 + 1\right\rangle \qquad 0 \leqslant u < \infty; \quad 0 \leqslant w \leqslant 2\pi$$

Here is a link to a list of surfaces.

One common example of a surface which does not have an obvious parametrization is a level surface, $f(x,y,z) = const$. There is a theorem, called the *Implicit Function Theorem* which says that a level surface $f(x,y,z) = const$ has a parametrization in a neighborhood of a point $\langle a,b,c\rangle$ provided one of the partial derivatives is non-zero at the point. This is useful for theoretical work since in deriving an equation we can usually assume a surface is parametrized even if we can't write a parametrization down.

1.1. **Grid lines.** In the picture of the Dini surface you can see two colors of greens and some lines called *grid lines*. These are curves obtained by setting $u = const.$ or $w = const.$

Each grid line is a curve. For example, if we fix $w = b$, then the parametrized curve is

$$\rho(t) = \langle x(t,b), y(t,b), z(t,b)\rangle$$

It turns out that the tangent lines to the grid lines are important. The calculation is easy. Define

$$\mathbf{r}_u(u,w) = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$
$$\mathbf{r}_w(u,w) = \left\langle \frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w} \right\rangle$$

- The vector $\mathbf{r}_u(u,w)$ evaluated at $(a,b)$ is a tangent vector to the grid line $\rho(t) = \langle x(t,b), y(t,b), z(t,b)\rangle$ at $t = a$.
- The vector $\mathbf{r}_w(u,w)$ evaluated at $(a,b)$ is a tangent vector to the grid line $\rho(t) = \langle x(a,t), y(a,t), z(a,t)\rangle$ at $t = b$.

*A vector function $\boldsymbol{r}(u,w)$ is a smooth surface provided $\boldsymbol{r}_u(u,w)$ and $\boldsymbol{r}_w(u,w)$ are continuous and not parallel at any point in $D$, except maybe along the boundary of $D$. This means that the image of $\boldsymbol{r}$ is 2 dimensional.*

One can also speak of a point the image of $\mathbf{r}(u,w)$ as being *smooth* which means that $\mathbf{r}_u(u,w)$ and $\mathbf{r}_w(u,w)$ are not parallel at the point.

The reason for the name "grid lines" comes from looking at Figure 5 from the book:

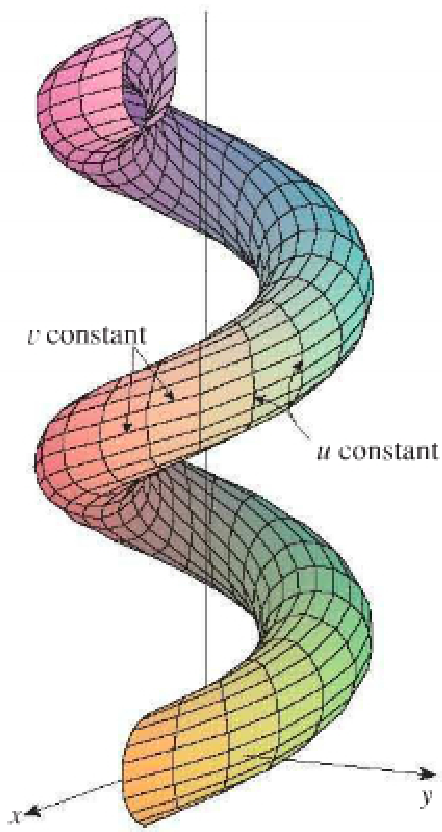
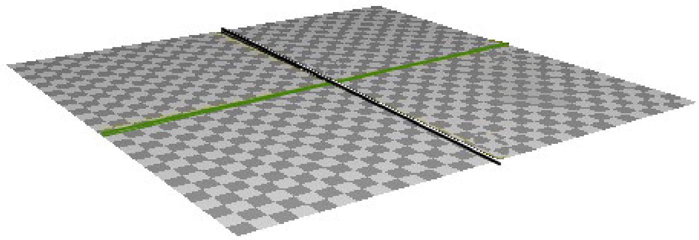



**FIGURE 5**

Figure 5 from the book

The surface in Figure 5 is given by

$$\mathbf{r}(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u + \cos v\rangle$$



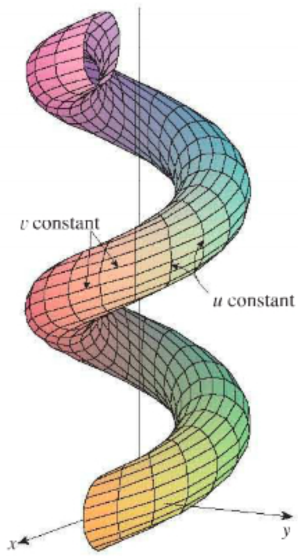
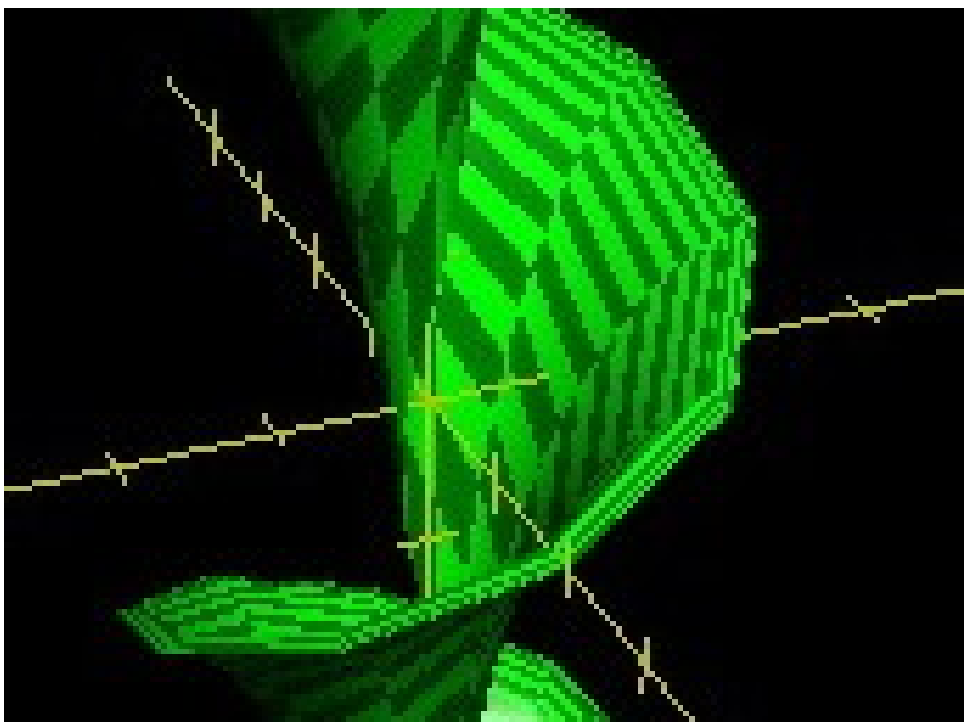


FIGURE 5

Each little square in the plane on the left goes over to a piece on the right. Each grid line in the plane goes over to one of the curves on the surface. Most graphics programs draw some grid lines for you. You can see some in the picture of the surface of Dini from above.



## 2. Tangent planes

At a smooth point on a surface, the *tangent plane* to the surface at the point is the plane which contains the point and the two vectors $\mathbf{r}_u(u, w)$ and $\mathbf{r}_w(u, w)$.

Then the normal vector to the tangent plane is

$$\mathbf{r}_u(u, w) \times \mathbf{r}_w(u, w)$$

so a point is smooth if and only if $\mathbf{r}_u(u, w) \times \mathbf{r}_w(u, w) \neq \langle 0, 0, 0\rangle$ or $|\mathbf{r}_u(u, w) \times \mathbf{r}_w(u, w)| \neq 0$.

The parametric equation for the tangent plane at the point $\mathbf{r}(a, b)$ is

$$\mathbf{r}(a, b) + u\mathbf{r}_u(a, b) + w\mathbf{r}_w(a, b)$$

OR

$$\big(\mathbf{r}_u(a, b) \times \mathbf{r}_w(a, b)\big) \bullet \langle x, y, z\rangle = \big(\mathbf{r}_u(a, b) \times \mathbf{r}_w(a, b)\big) \bullet \mathbf{r}(a, b)$$

The *normal line* to a surface through a point on the surface is the line perpendicular to the tangent plane. If the point is $\mathbf{r}(a, b)$, the equation is

$$\mathbf{r}(a, b) + \big(\mathbf{r}_u(a, b) \times \mathbf{r}_w(a, b)\big)t$$

**Example.** The sphere of radius $a > 0$ centered at the origin. One description is the level surface $x^2 + y^2 + z^2 = a^2$. A normal vector is the gradient $\langle 2x, 2y, 2z\rangle$. A unit normal vector is

$$\mathbf{n}_1(x, y, z) = \left\langle \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right\rangle$$

A parametrization of the upper half of this sphere is given by

$$\mathbf{r}(u, w) = \left\langle u, w, \sqrt{a^2 - u^2 - w^2} \right\rangle; \qquad u^2 + w^2 \leqslant a^2$$

$$\begin{aligned}
\mathbf{r}_u &= \left\langle 1, 0, \frac{-u}{\sqrt{a^2 - u^2 - w^2}} \right\rangle \\
\mathbf{r}_w &= \left\langle 0, 1, \frac{-w}{\sqrt{a^2 - u^2 - w^2}} \right\rangle \\
\mathbf{r}_u \times \mathbf{r}_w &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{-u}{\sqrt{a^2 - u^2 - w^2}} \\ 0 & 1 & \frac{-w}{\sqrt{a^2 - u^2 - w^2}} \end{vmatrix} = \left\langle \frac{u}{\sqrt{a^2 - u^2 - w^2}}, \frac{w}{\sqrt{a^2 - u^2 - w^2}}, 1 \right\rangle \\
|\mathbf{r}_u \times \mathbf{r}_w| &= \frac{a}{\sqrt{a^2 - u^2 - w^2}}
\end{aligned}$$

A unit normal vector is

$$\mathbf{n}_2(u, w) = \left\langle \frac{u}{a}, \frac{w}{a}, \frac{\sqrt{a^2 - u^2 - w^2}}{a} \right\rangle$$

This unit normal vector at a given point is equal to the unit normal we got from the gradient. At any point $(x_0, y_0, z_0)$ with $z_0 > 0$, $\mathbf{r}(x_0, y_0) = \langle x_0, y_0, z_0\rangle$. Then

$$\mathbf{n}_2(x_0, y_0) = \left\langle \frac{x_0}{a}, \frac{y_0}{a}, \frac{\sqrt{a^2 - x_0^2 - y_0^2}}{a} \right\rangle = \left\langle \frac{x_0}{a}, \frac{y_0}{a}, \frac{z_0}{a} \right\rangle = \mathbf{n}_1(x_0, y_0, z_0)$$

We can parametrize the sphere using spherical coordinates.

$$\mathbf{r}(\theta, \phi) = \langle a\cos(\theta)\sin(\phi), a\sin(\theta)\sin(\phi), a\cos(\phi)\rangle; \qquad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi$$

$$
\begin{aligned}
\mathbf{r}_\theta &= \langle -a\sin(\theta)\sin(\phi), a\cos(\theta)\sin(\phi), 0\rangle \\
\mathbf{r}_\phi &= \langle a\cos(\theta)\cos(\phi), a\sin(\theta)\cos(\phi), -a\sin(\phi)\rangle \\
\mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(\theta)\sin(\phi) & a\cos(\theta)\sin(\phi) & 0 \\ a\cos(\theta)\cos(\phi) & a\sin(\theta)\cos(\phi) & -a\sin(\phi) \end{vmatrix} = \\
&\left\langle -a^2\cos(\theta)\sin^2(\phi), -a^2\sin(\theta)\sin^2(\phi), -a^2\sin(\phi)\cos(\phi)\right\rangle = \\
&-a^2\sin(\phi)\left\langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\right\rangle \\
|\mathbf{r}_\theta \times \mathbf{r}_\phi| &= a^2\sin(\phi)
\end{aligned}
$$

A unit normal vector is

$$\mathbf{n}_3(\theta,\phi) = -\left\langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\right\rangle$$

This unit normal vector at a given point is the negative of the unit normal we got from the gradient. At any point $(x_0, y_0, z_0)$ find $\theta_0$ and $\phi_0$ so that $\mathbf{r}(\theta_0,\phi_0) = \langle x_0, y_0, z_0\rangle$. The parametrization guarantees that this can be done. Then

$$
\begin{aligned}
&\mathbf{n}_3(\theta_0,\phi_0) = -\left\langle \cos(\theta_0)\sin(\phi_0), \sin(\theta_0)\sin(\phi_0), \cos(\phi_0)\right\rangle = \\
&\quad -\left\langle \frac{a\cos(\theta_0)\sin(\phi_0)}{a}, \frac{a\sin(\theta_0)\sin(\phi_0)}{a}, \frac{a\cos(\phi_0)}{a}\right\rangle = -\left\langle \frac{x_0}{a}, \frac{y_0}{a}, \frac{z_0}{a}\right\rangle = -\mathbf{n}_1(x_0,y_0,z_0)
\end{aligned}
$$

If we parametrize the sphere using cylindrical coordinates,

$$\mathbf{r}(\theta,z) = \left\langle \sqrt{a^2-z^2}\cos(\theta), \sqrt{a^2-z^2}\sin(\theta), z\right\rangle; \qquad 0 \leqslant \theta \leqslant 2\pi, \quad -a \leqslant z \leqslant a$$

$$
\begin{aligned}
\mathbf{r}_\theta &= \left\langle -\left(\sqrt{a^2-z^2}\right)\sin(\theta), \left(\sqrt{a^2-z^2}\right)\cos(\theta), 0\right\rangle \\
\mathbf{r}_z &= \left\langle -\frac{z}{\sqrt{a^2-z^2}}\cos(\theta), -\frac{z}{\sqrt{a^2-z^2}}\sin(\theta), 1\right\rangle \\
\mathbf{r}_\theta \times \mathbf{r}_z &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\left(\sqrt{a^2-z^2}\right)\sin(\theta) & \left(\sqrt{a^2-z^2}\right)\cos(\theta) & 0 \\ -\frac{z}{\sqrt{a^2-z^2}}\cos(\theta) & -\frac{z}{\sqrt{a^2-z^2}}\sin(\theta) & 1 \end{vmatrix} = \left\langle \left(\sqrt{a^2-z^2}\right)\cos(\theta), \left(\sqrt{a^2-z^2}\right)\sin(\theta), z\right\rangle
\end{aligned}
$$

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = a$$

A unit normal vector is

$$\mathbf{n}_4(\theta,\phi) = \left\langle \frac{\sqrt{a^2-z^2}}{a}\cos(\theta), \frac{\sqrt{a^2-z^2}}{a}\sin(\theta), \frac{z}{a}\right\rangle$$

This unit normal vector at a given point is equal to the unit normal we got from the gradient. At any point $(x_0, y_0, z_0)$ find $\theta_0$ so that $\mathbf{r}(\theta_0, z_0) = \langle x_0, y_0, z_0\rangle$. Then

$$\mathbf{n}_4(\theta_0,z_0) = \left\langle \frac{\sqrt{a^2-z_0^2}}{a}\cos(\theta_0), \frac{\sqrt{a^2-z_0^2}}{a}\sin(\theta_0), \frac{z_0}{a}\right\rangle = \left\langle \frac{x_0}{a}, \frac{y_0}{a}, \frac{z_0}{a}\right\rangle = \mathbf{n}_1(x_0,y_0,z_0)$$

## 3. Orientation

An *orientation* for a surface is a continuous choice of unit normal vector for the surface. Every surface which can be parametrized has two choices of a unit normal vector at every point.

The boundary of a solid $E$ in three space has orientations. By $\partial E$ we mean the boundary of $E$ with the orientation for which the normal vector points out of $E$.

**Example:** If $E$ is the ball of radius $a$ centered at the origin then $\partial E$ is the sphere of radius $a$ centered at the origin. The gradient vector of the level curve $x^2 + y^2 + z^2 = a^2$, namely $2\langle x, y, z\rangle$ points out. Hence the normal vectors we computed in rectangular and cylindrical coordinates point out, and the normal vector we computed in spherical coordinates points inward.

An important skill is to be able to decide if a given continuous normal field on the boundary of a solid is outward or inward. To do this, pick a point on the boundary. The easiest way to do this is to pick a point in the parameter space. Then compute the normal vector you calculated at that point. Then check if it points the way you want.

**Example:** The sphere bounds the ball. Let us use spherical coordinates so

$$\mathbf{r}(\theta,\phi) = \langle a\cos(\theta)\sin(\phi), a\sin(\theta)\sin(\phi), a\cos(\phi)\rangle$$

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = -a^2\sin(\phi)\langle\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle$$

Pick a point, say $\left(0, \frac{\pi}{2}\right)$ in $\theta\phi$ space. Then $\mathbf{r}\left(0, \frac{\pi}{2}\right) = \langle a, 0, 0\rangle$. At this point $\mathbf{r}_\theta \times \mathbf{r}_\phi\left(0, \frac{\pi}{2}\right) = -a^2\langle 1, 0, 0\rangle$ so the normal vector points inward. If the outward normal is required, which it often is, just use $a^2\sin(\phi)\langle\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle$.

The moral of this last example is to calculate one normal field and check at a convenient point whether it is the one you need. If it is not, the negative of it will be.

Surprisingly, not all surfaces are orientable. The German mathematician Möbius realized that the Möbius strip was such a surface. One parametrization is

$$\mathbf{r}(u,w) = \left\langle \left(a + bw\cos\left(\frac{u}{2}\right)\right)\cos(u), \left(a + bw\cos\left(\frac{u}{2}\right))\right)\sin(u), bw\sin\left(\frac{u}{2}\right)\right\rangle; \quad 0 \leqslant u \leqslant 2\pi,\ -1 \leqslant w \leqslant 1$$

with $a > b > 0$.

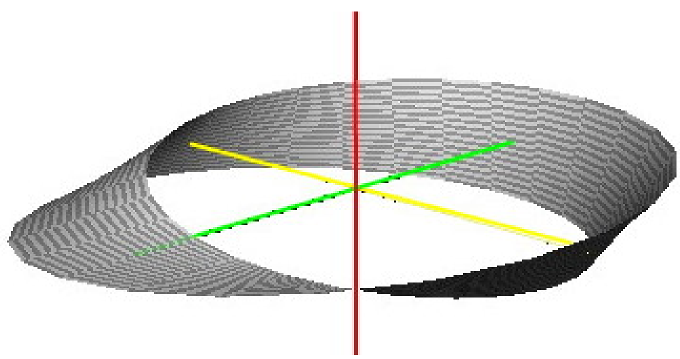
If

$$\mathbf{A}(u) = \langle \cos(u), \sin(u), 0\rangle$$

then

$$\mathbf{r}(u,w) = \left(a + bw\cos\left(\frac{u}{2}\right)\right)\mathbf{A}(u) + bw\sin\left(\frac{u}{2}\right)\mathbf{k}$$

and $\mathbf{A}(u)$ has $z$-coordinate 0. Note $\mathbf{A}$ has length 1 (as of course does $\mathbf{k}$) and $\mathbf{A} \bullet \mathbf{k} = 0$.



$a = 1$ and $b = 0.5$

The $z$ axis is red, the $x$ axis is green and the $y$ axis is yellow.

The parametrization is "mostly" one-to-one. Note first that since $a > b$, $a + bw\cos\left(\frac{u}{2}\right) > 0$. If $\mathbf{r}(u_1, w_1) = \mathbf{r}(u_0, w_0)$ we may also assume $u_0 \leqslant u_1$. Then $A(u_1) = A(u_0)$, $w_1\sin\left(\frac{u_1}{2}\right) = w_0\sin\left(\frac{u_0}{2}\right)$ and $a + bw_1\cos\left(\frac{u_1}{2}\right) = a + bw_0\cos\left(\frac{u_0}{2}\right)$.

From $A(u_1) = A(u_0)$ it follows that $u_1 = u_0$ or $u_0 = 0$ and $u_1 = \pi$.

If $u_0 = 0$, $u_1 = \pi$ it follows from $w_1\sin\left(\frac{u_1}{2}\right) = w_0\sin\left(\frac{u_0}{2}\right)$ that $w_1 = -w_0$.

Hereafter we need only discuss the case $u_1 = u_1$. From $a + bw_0\cos\left(\frac{u_0}{2}\right) = a + bw_1\cos\left(\frac{u_1}{2}\right)$ it follows that $w_1\cos\left(\frac{u_1}{2}\right) = w_0\cos\left(\frac{u_0}{2}\right)$ and since $w_1\sin\left(\frac{u_1}{2}\right) = w_0\sin\left(\frac{u_0}{2}\right)$, $w_1 = w_0$.

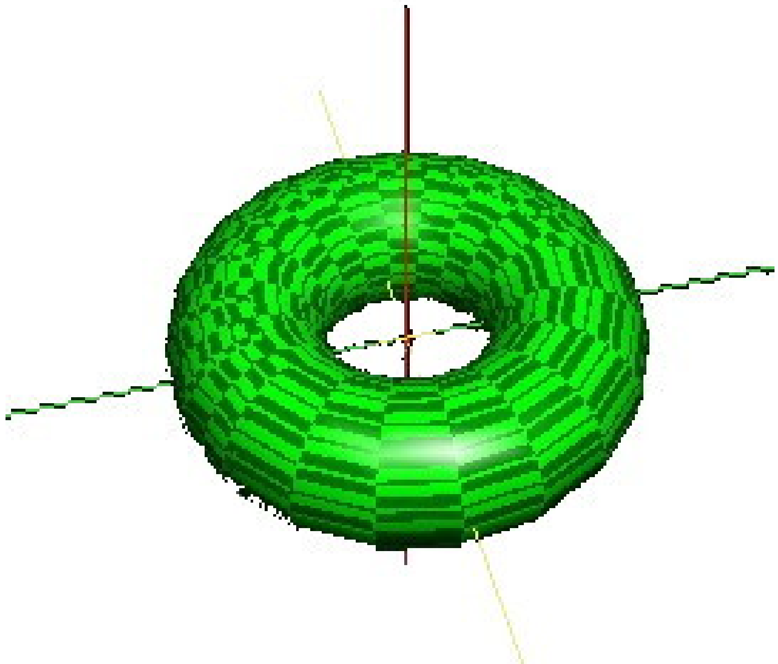
Hence $\mathbf{r}$ is one-to-one except for

$$\mathbf{r}(0,w) = \mathbf{r}(2\pi, -w)$$

**Example.** A torus

$$\mathbf{r}(u,w) = \left\langle \big(a + b\cos(w)\big)\cos(u), \big(a + b\cos(w)\big)\sin(u), b\sin(w)\right\rangle; \qquad 0 \leqslant u \leqslant 2\pi, \quad 0 \leqslant w \leqslant 2\pi$$

with $a > b > 0$.



Torus is $a = 2$, $b = 1$

Note

$$\mathbf{r}_u(u,w) = \big\langle -\big(a + b\cos(w)\big)\sin(u), \big(a + b\cos(w)\big)\cos(u), 0\big\rangle$$
$$\mathbf{r}_w(u,w) = \langle -b\sin(w)\cos(u), -b\sin(w)\sin(u), b\cos(w)\rangle$$
$$\mathbf{r}_u \times \mathbf{r}_w = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\big(a + b\cos(w)\big)\sin(u) & \big(a + b\cos(w)\big)\cos(u) & 0 \\ -b\sin(w)\cos(u) & -b\sin(w)\sin(u) & b\cos(w) \end{vmatrix} =$$
$$\Big\langle b\cos(w)\cos(u)\big(a + b\cos(w)\big), b\cos(w)\sin(u)\big(a + b\cos(w)\big), b\sin(w)\big(a + b\cos(w)\big)\Big\rangle =$$

$$\mathbf{r}_u \times \mathbf{r}_w = b\big(a + b\cos(w)\big)\,\langle\cos(w)\cos(u), \cos(w)\sin(u), \sin(w)\rangle$$
$$|\mathbf{r}_u \times \mathbf{r}_w| = b\big(a + b\cos(w)\big)$$

The torus is the boundary of a solid and so is orientable. At $(0,0)$ the point is $\mathbf{r}(0,0) = \langle a+b, 0, 0\rangle$ which is the largest of the four points on the $x$-axis. (The other three are $\langle a-b, 0, 0\rangle$, $\langle -a+b, 0, 0\rangle$ and $\langle -a-b, 0, 0\rangle$.)

At $(0,0)$, $\mathbf{r}_u \times \mathbf{r}_w(0,0) = b(a+b)\,\langle 1, 0, 0\rangle$ and since $b(a+b) > 0$, this normal field points out.

There is no need to check additional points but suppose you were asked to check the direction at $\langle a-b, 0, 0\rangle$. Now the first problem is to find $(u,w)$ so that $\mathbf{r}(u,w) = \langle a-b, 0, 0\rangle$. To get the $z$-coordinate right, $w = 0$ or $\pi$ or $2\pi$. $w = 0$ or $2\pi$ gives an $a+b$ factor in the other two coordinates so $w = \pi$ looks like the right way to go. To get the $y$-coordinate right, take $u = 0$ and we see $\mathbf{r}(0,\pi) = \langle a-b, 0, 0\rangle$. At this point, $\mathbf{r}_u \times \mathbf{r}_w(0,\pi) = a(a-b)\,\langle -1, 0, 0\rangle$. Since $a(a-b) > 0$, this points towards the origin, which is outward from the solid!

## 4. Surface integrals

Let $T$ be a surface in $\mathbb{R}^3$. Let $f\colon T \to \mathbb{R}$ be a function defined on $T$. Define

$$\iint\limits_T f\,dS = \lim_{mesh(\mathcal{P})\to 0} \sum_{\mathcal{P}} f(\mathbf{p}_i)\text{Area}(T_i)$$

as a limit of Riemann sums over sampled-partitions. A *sampled-partition of* $T$, $\mathcal{P}$, is a division of the surface $T$ into pieces, $T_i$, followed by a choice of *sample point*, $\mathbf{p}_i$, in each $T_i$. By a division of a surface into pieces we mean $\cup_i T_i = T$ and $T_i \cap T_j$ lies in the boundary of each piece.

The associated Riemann sum is $\sum_i f(\mathbf{p}_i)\text{Area}(T_i)$. The *mesh* of the sampled-partition is $\leq \epsilon$ provided every $T_i$ is contained in the ball of radius $\epsilon$ centered at $\mathbf{p}_i$.

If $f$ is continuous, $\displaystyle\iint_T f\,dS$ is well-defined, and hence a number.

- $\displaystyle\iint_T 1\,dS$ is the area.
- $\displaystyle\iint_T \mu\,dS$ is the mass if $\mu$ is the density.
- $\displaystyle\iint_T q\,dS$ is the total charge if $q$ is the charge density.
- $\displaystyle\iint_T z\mu\,dS$ is the moment about the $M_{xy}$ plane where $\mu$ is the density. $\bar{z} = \dfrac{\iint_T z\mu\,dS}{\iint_T \mu\,dS}$.
- $\displaystyle\iint_T x\mu\,dS$ is the moment about the $M_{yz}$ plane where $\mu$ is the density. $\bar{x} = \dfrac{\iint_T x\mu\,dS}{\iint_T \mu\,dS}$.
- $\displaystyle\iint_T y\mu\,dS$ is the moment about the $M_{xz}$ plane where $\mu$ is the density. $\bar{y} = \dfrac{\iint_T y\mu\,dS}{\iint_T \mu\,dS}$.

I'll leave it to you to work out the formulas for moments of inertia.

## 5. Surface integrals as double integrals

Parametrize $S$ as $\mathbf{r}(u, w)\colon D \to \mathbb{R}^3$ where $D$ is some bounded region in the plane. We can evaluate $f$ at points in $S$ as $f\big(\mathbf{r}(u, w)\big)$. The remaining issue is to figure out $\text{Area}(T_i)$.

At a point $\mathbf{r}(u_0, w_0)$ we can look at the tangent plane to $S$ at the point. It is

$$\mathbf{r}(u_0, w_0) + r\mathbf{r}_u(u_0, w_0) + t\mathbf{r}_w(u_0, w_0)$$

In the tangent plane we have the parallelogram with vertices $\mathbf{r}(u_0, w_0)$, $\mathbf{r}(u_0, w_0) + \mathbf{r}_u(u_0, w_0)\,du$, $\mathbf{r}(u_0, w_0) + \mathbf{r}_w(u_0, w_0)\,dw$ and $\mathbf{r}(u_0, w_0) + (\mathbf{r}_u(u_0, w_0)\,du + \mathbf{r}_w(u_0, w_0)\,dw)$. The area of this parallelogram is $|\mathbf{r}_u(u_0, w_0) \times \mathbf{r}_w(u_0, w_0)|\,du\,dw = |\mathbf{r}_u(u_0, w_0) \times \mathbf{r}_w(u_0, w_0)|\,dA$ and we use this as the approximation to $\text{Area}(T_i)$ and

$$(*) \qquad \iint_T f\,dS = \iint_D f\big(\mathbf{r}(u, w)\big)\,|\mathbf{r}_u \times \mathbf{r}_w|\,dA$$

**Remark:** If the third coordinate of $\mathbf{r}$ is 0, note $|\mathbf{r}_u(u_0, w_0) \times \mathbf{r}_w(u_0, w_0)|$ is just the absolute value of the Jacobian of the coordinate transformation.

## 6. Area examples

**Surface area of a sphere 1:** The upper hemisphere was parametrized above and we saw $|\mathbf{r}_u \times \mathbf{r}_w| = \dfrac{a}{\sqrt{a^2-u^2-w^2}}$ with $u^2+w^2 \leqslant a^2$. Hence

$$\text{Surface area} = \iint\limits_{u^2+w^2\leqslant a^2} \frac{a}{\sqrt{a^2-u^2-w^2}}\,dA = \int_0^a \int_0^{2\pi} \frac{ar}{\sqrt{a^2-r^2}}\,d\theta\,dr = -2\pi a\sqrt{a^2-r^2}\,\Big|_0^a = 2\pi a^2$$

The surface area of the full sphere is $4\pi a^2$.

**Surface area of a sphere 2:** Using spherical coordinates, $|\mathbf{r}_\theta \times \mathbf{r}_\phi| = a^2\sin(\phi)$ with $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \pi$. Hence

$$\text{Surface area} = \iint\limits_{\substack{0\leqslant\theta\leqslant 2\pi\\ 0\leqslant\phi\leqslant\pi}} a^2\sin(\phi)\,dA = \int_0^\pi \int_0^{2\pi} a^2\sin(\phi)\,d\theta\,d\phi = -2\pi a^2\cos(\phi)\,\Big|_0^\pi = 4\pi a^2$$

**Surface area of a sphere 3:** Using cylindrical coordinates, $|\mathbf{r}_\theta \times \mathbf{r}_z| = a$ with $0 \leqslant \theta \leqslant 2\pi$, $-a \leqslant z \leqslant a$. Hence

$$\text{Surface area} = \iint\limits_{\substack{0\leqslant\theta\leqslant 2\pi\\ -a\leqslant z\leqslant a}} a\,dA = \int_{-a}^a \int_0^{2\pi} a\,dz\,d\theta = a(2a)(2\pi) = 4\pi a^2$$

**Surface area of a torus:** $|\mathbf{r}_u \times \mathbf{r}_w| = b\big(a+b\cos(w)\big)$ with $0 \leqslant u \leqslant 2\pi$, $0 \leqslant w \leqslant 2\pi$.
Hence

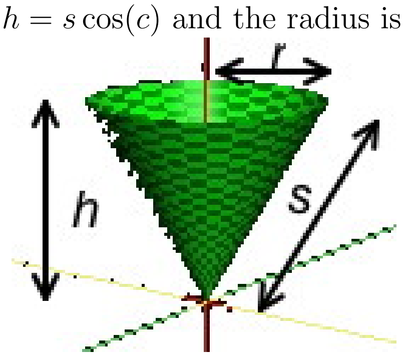
$$\text{Surface area} = \iint\limits_{\substack{0\leqslant u\leqslant 2\pi\\ 0\leqslant w\leqslant 2\pi}} b\big(a+b\cos(w)\big)\,dA = \int_0^{2\pi}\int_0^{2\pi} b\big(a+b\cos(w)\big)\,dw\,du =$$

$$b\int_0^{2\pi} aw + b\sin(w)\Big|_0^{2\pi}\,du = b\int_0^{2\pi} 2\pi a\,du = 4\pi^2 ab$$

**Surface area of a cone:** Parametrize the cone in spherical coordinates as

$$\mathbf{r}(\rho,\theta) = \langle \rho\cos(\theta)\sin(c), \rho\sin(\theta)\sin(c), \rho\cos(c)\rangle$$

with $0 \leqslant \rho \leqslant s$, $0 \leqslant \theta \leqslant 2\pi$. Here $s$ is the slant height of the cone and $c$ is the angle the cone makes down from the $z$-axis. The height is $h = s\cos(c)$ and the radius is $r = s\sin(c)$.




Cone with $s = 3$, $c = \frac{\pi}{3}$

$$\begin{aligned}
\mathbf{r}_\rho &= \langle \cos(\theta)\sin(c), \sin(\theta)\sin(c), \cos(c)\rangle \\
\mathbf{r}_\theta &= \langle -\rho\sin(\theta)\sin(c), \rho\cos(\theta)\sin(c), 0\rangle \\
\mathbf{r}_\rho \times \mathbf{r}_\theta &= \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta)\sin(c) & \sin(\theta)\sin(c) & \cos(c) \\ -\rho\sin(\theta)\sin(c) & \rho\cos(\theta)\sin(c) & 0 \end{vmatrix} = \rho\left\langle \cos(\theta)\sin(c)\cos(c), \sin(\theta)\sin(c)\cos(c), \sin^2(c)\right\rangle
\end{aligned}$$

$$|\mathbf{r}_\rho \times \mathbf{r}_\theta| = \rho\sin(c)$$

Hence

$$\text{Surface area} = \iint\limits_{\substack{0\leqslant\rho\leqslant s\\0\leqslant\theta\leqslant 2\pi}} \rho\sin(c)\,dA = \int_0^s\int_0^{2\pi} \rho\sin(c)\,d\theta\,d\rho = 2\pi\sin(c)\int_0^s \rho\,d\rho = \pi s^2\sin(c) = \pi s r = \pi r h\sec(c)$$

**Surface area of a cone:** Parametrize the cone in cylindrical coordinates.

$$\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r\cot(c)\rangle$$

with $0 \leqslant r \leqslant R$, $0 \leqslant \theta \leqslant 2\pi$.

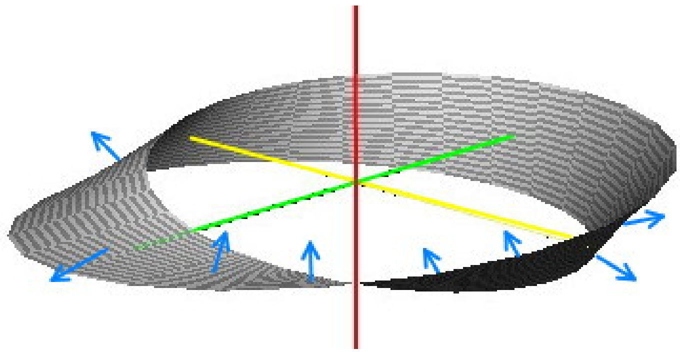
$$\begin{aligned}
\mathbf{r}_r &= \langle \cos(\theta), \sin(\theta), \cot(c)\rangle \\
\mathbf{r}_\theta &= \langle -r\sin(\theta), r\cos(\theta), 0\rangle \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & \cot(c) \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle -r\cos(\theta)\cot(c), -r\sin(\theta)\cot(c), r\rangle \\
|\mathbf{r}_\rho \times \mathbf{r}_\theta| &= \sqrt{r^2\cot^2(c) + r^2} = \sqrt{r^2\csc^2(c)} = r\csc(c)
\end{aligned}$$

Hence

$$\text{Surface area} = \iint\limits_{\substack{0\leqslant r\leqslant R\\0\leqslant\theta\leqslant 2\pi}} r\csc(c)\,dA = \int_0^R\int_0^{2\pi} r\csc(c)\,d\theta\,d\rho = 2\pi\csc(c)\int_0^R r\,dr = \pi R^2\csc(c) = \pi s R = \pi R h\sec c$$

## 7. Appendix: More about Möbius strip

Here is a picture with some normal vectors drawn.



Notice that as you move around the Möbius strip starting with the outer right side you end up with the normal pointing the other way.

To compute normal vectors, recall

$$\mathbf{r}(u,w) = \left(a + bw\cos\left(\frac{u}{2}\right)\right)\mathbf{A}(u) + bw\sin\left(\frac{u}{2}\right)\mathbf{k}$$

$$\begin{aligned}
\mathbf{r}_u &= -\frac{bw}{2}\sin\left(\frac{u}{2}\right)\mathbf{A}(u) + \left(a + bw\cos\left(\frac{u}{2}\right)\right)\mathbf{A}_u(u) + \frac{bw}{2}\cos\left(\frac{u}{2}\right)\mathbf{k} \\
\mathbf{r}_w &= \frac{b}{2}\cos\left(\frac{u}{2}\right)\mathbf{A}(u) + \frac{b}{2}\sin\left(\frac{u}{2}\right)\mathbf{k}
\end{aligned}$$

Let

$$\mathbf{A}_u(u) = \langle -\sin(u), \cos(u), 0\rangle = \mathbf{B}(u)$$

Note $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{k}$ are all unit length and mutually orthogonal. Moreover

$$\begin{array}{rcccccccc}
\mathbf{r}_u & = & -\frac{bw}{2}\sin\left(\frac{u}{2}\right)\mathbf{A}(u) & + & \left(a + bw\cos\left(\frac{u}{2}\right)\right) & \mathbf{B}(u) & + & \frac{bw}{2}\cos\left(\frac{u}{2}\right)\mathbf{k} \\
\mathbf{r}_w & = & \frac{b}{2}\cos\left(\frac{u}{2}\right)\mathbf{A}(u) & + & 0 & \mathbf{B}(u) & + & \frac{b}{2}\sin\left(\frac{u}{2}\right)\mathbf{k}
\end{array}$$

$$\mathbf{A}(u)\times\mathbf{k} = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(u) & \sin(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \sin(u), -\cos(u), 0\rangle = -\mathbf{B}(u)$$

$$\mathbf{A}(u)\times\mathbf{B}(u) = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(u) & \sin(u) & 0 \\ -\sin(u) & \cos(u) & 0 \end{vmatrix} = \mathbf{k}$$

$$\mathbf{B}(u)\times\mathbf{k} = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos(u), \sin(u), 0\rangle = \mathbf{A}(u)$$

$$\mathbf{r}_u \bullet \mathbf{r}_w = 0$$

It is a good exercise for you to verify that

$$\mathbf{r}_u \times \mathbf{r}_w = \det \begin{vmatrix} \mathbf{A} & \mathbf{B} & \mathbf{k} \\ -\frac{bw}{2}\sin\left(\frac{u}{2}\right) & \left(a + bw\cos\left(\frac{u}{2}\right)\right) & \frac{bw}{2}\cos\left(\frac{u}{2}\right) \\ \frac{b}{2}\cos\left(\frac{u}{2}\right) & 0 & \frac{b}{2}\sin\left(\frac{u}{2}\right) \end{vmatrix}$$

$$\mathbf{r}_u \times \mathbf{r}_w = \frac{b}{2}\sin\left(\frac{u}{2}\right)\left(a + bw\cos\left(\frac{u}{2}\right)\right)\mathbf{A}(u) + \frac{b^2 w}{4}\mathbf{B}(u) - \frac{b}{2}\cos\left(\frac{u}{2}\right)\left(a + bw\cos\left(\frac{u}{2}\right)\right)\mathbf{k}$$

Hence

$$|\mathbf{r}_u \times \mathbf{r}_w|^2 = \frac{b^2}{4}\sin^2\left(\frac{u}{2}\right)\left(a + bw\cos\left(\frac{u}{2}\right)\right)^2 + \frac{b^4 w^2}{16} + \frac{b^2}{4}\cos^2\left(\frac{u}{2}\right)\left(a + bw\cos\left(\frac{u}{2}\right)\right)^2 =$$

$$\frac{b^2}{4}\left(a + bw\cos\left(\frac{u}{2}\right)\right)^2 + \frac{b^4 w^2}{16}$$

Therefore the surface area is

$$\frac{b}{2}\int_0^{2\pi}\int_{-1}^{1}\sqrt{\left(a + bw\cos\left(\frac{u}{2}\right)\right)^2 + \frac{b^2 w^2}{4}}\,dw\,du$$

By substitution and algebra this is

$$\frac{a^2}{2}\int_0^{2\pi}\int_{-\frac{b}{a}}^{\frac{b}{a}}\sqrt{\left(1 + v\cos\left(\frac{u}{2}\right)\right)^2 + \frac{v^2}{4}}\,dv\,du$$

To see the non-orientable nature of the Möbius strip, consider the grid line $w = 0$. The curve is the circle of radius $a$ centered at the origin in the $xy$ plane. Note $\mathbf{r}(0,0) = \mathbf{r}(2\pi, 0)$, but

$$\mathbf{r}_u \times \mathbf{r}_w(0,0) = -\frac{ab}{2}\langle 0,0,1\rangle$$
$$\mathbf{r}_u \times \mathbf{r}_w(0,2\pi) = \frac{ab}{2}\langle 0,0,1\rangle$$

The full level curve at $w = 0$ is $\mathbf{c}(t) = a\mathbf{A}(t)$, $0 \leqslant t \leqslant 2\pi$. The normal vector field at the point $\mathbf{c}(t)$ is

$$\mathbf{n}(t) = \frac{ab}{2}\sin\left(\frac{t}{2}\right)\mathbf{A}(t) - \frac{ab}{2}\cos\left(\frac{t}{2}\right)\mathbf{k}; \qquad 0 \leqslant t \leqslant 2\pi$$

On the interval $[0, \pi)$, $\mathbf{n}(t)$ points up and on $(\pi, 2\pi]$ it points down. It is the fact that $\mathbf{r}(0,0) = \mathbf{r}(2\pi, 0)$ that prevents the Möbius strip from having a continuous normal field.

## 8. Gauss curvature

The great German mathematician Gauss worked out what the analogue of the curvature of a surface ought to be. At each point on the surface the curvature is a number which measures how "curved" the surface is.

$$K = \frac{\det \begin{vmatrix} \mathbf{r}_{uu} \bullet (\mathbf{r}_u \times \mathbf{r}_w) & \mathbf{r}_{uw} \bullet (\mathbf{r}_u \times \mathbf{r}_w) \\ \mathbf{r}_{wu} \bullet (\mathbf{r}_u \times \mathbf{r}_w) & \mathbf{r}_{ww} \bullet (\mathbf{r}_u \times \mathbf{r}_w) \end{vmatrix}}{|\mathbf{r}_u \times \mathbf{r}_w|^4}$$

$\mathbf{r}(u,w) = \left\langle u, w, \sqrt{a^2-u^2-w^2} \right\rangle$, $a > 0$.

$$\mathbf{r}_u = \left\langle 1, 0, \frac{-u}{\sqrt{a^2-u^2-w^2}} \right\rangle \ \mathbf{r}_w = \left\langle 0, 1, \frac{-w}{\sqrt{a^2-u^2-w^2}} \right\rangle$$

$$\mathbf{r}_u \times \mathbf{r}_w = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{-u}{\sqrt{a^2-u^2-w^2}} \\ 0 & 1 & \dfrac{-w}{\sqrt{a^2-u^2-w^2}} \end{vmatrix} = \left\langle \frac{u}{\sqrt{a^2-u^2-w^2}}, \frac{w}{\sqrt{a^2-u^2-w^2}}, 1 \right\rangle$$

$$|\mathbf{r}_u \times \mathbf{r}_w| = \frac{a}{\sqrt{a^2-u^2-w^2}}$$

$$\mathbf{r}_{uu} = \left\langle 0, 0, \frac{-1}{\sqrt{a^2-u^2-w^2}} - \frac{u^2}{\left(\sqrt{a^2-u^2-w^2}\right)^3} \right\rangle = \left\langle 0, 0, -\frac{a^2-w^2}{\left(\sqrt{a^2-u^2-w^2}\right)^3} \right\rangle$$

$$\mathbf{r}_{uw} = \left\langle 0, 0, -\frac{uw}{\left(\sqrt{a^2-u^2-w^2}\right)^3} \right\rangle$$

$$\mathbf{r}_{ww} = \left\langle 0, 0, -\frac{a^2-u^2}{\left(\sqrt{a^2-u^2-w^2}\right)^3} \right\rangle$$

$$K = \frac{\det \begin{vmatrix} -\dfrac{a^2-w^2}{\left(\sqrt{a^2-u^2-w^2}\right)^3} & -\dfrac{uw}{\left(\sqrt{a^2-u^2-w^2}\right)^3} \\ -\dfrac{uw}{\left(\sqrt{a^2-u^2-w^2}\right)^3} & -\dfrac{a^2-u^2}{\left(\sqrt{a^2-u^2-w^2}\right)^3} \end{vmatrix}}{\dfrac{a^4}{(a^2-u^2-w^2)^2}} = \frac{\dfrac{(a^2-w^2)(a^2-u^2)-u^2w^2}{(a^2-u^2-w^2)^3}}{\dfrac{a^4}{(a^2-u^2-w^2)^2}} =$$

$$\frac{(a^4-a^2u^2-a^2w^2+u^2w^2)-u^2w^2}{a^4(a^2-u^2-w^2)} = \frac{a^4-a^2(u^2+w^2)}{a^4(a^2-u^2-w^2)} = \frac{1}{a^2}$$

In spherical coordinates:

$$\begin{aligned}
\mathbf{r}_\theta &= \langle -a\sin(\theta)\sin(\phi), a\cos(\theta)\sin(\phi), 0\rangle \\
\mathbf{r}_\phi &= \langle a\cos(\theta)\cos(\phi), a\sin(\theta)\cos(\phi), -a\sin(\phi)\rangle \\
\mathbf{r}_\theta \times \mathbf{r}_\phi &= -a^2\sin(\phi)\,\langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle \\
|\mathbf{r}_\theta \times \mathbf{r}_\phi| &= a^2\sin(\phi)
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_{\theta\theta} &= \langle -a\cos(\theta)\sin(\phi), -a\sin(\theta)\sin(\phi), 0\rangle = -a\sin(\phi)\,\langle\cos(\theta), \sin(\theta), 0\rangle \\
\mathbf{r}_{\theta\phi} &= \langle -a\sin(\theta)\cos(\phi), a\cos(\theta)\cos(\phi), 0\rangle = -a\cos(\phi)\,\langle\sin(\theta), -\cos(\theta), 0\rangle \\
\mathbf{r}_{\phi\phi} &= \langle -a\cos(\theta)\sin(\phi), -a\sin(\theta)\sin(\phi), -a\cos(\phi)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_{\theta\theta} \bullet (\mathbf{r}_\theta \times \mathbf{r}_\phi) &= -a^3 \sin^3(\phi) \\
\mathbf{r}_{\theta\phi} \bullet (\mathbf{r}_\theta \times \mathbf{r}_\phi) &= 0 \\
\mathbf{r}_{\phi\phi} \bullet (\mathbf{r}_\theta \times \mathbf{r}_\phi) &= -a^3 \sin(\phi)
\end{aligned}$$

so

$$K = \frac{a^6 \sin^4(\phi)}{\left(a^2 \sin(\phi)\right)^4} = \frac{1}{a^2}$$

It is a very unpleasant calculation to see that the surface of Dini given above has curvature $\dfrac{1}{2^2 + (0.4)^2}$