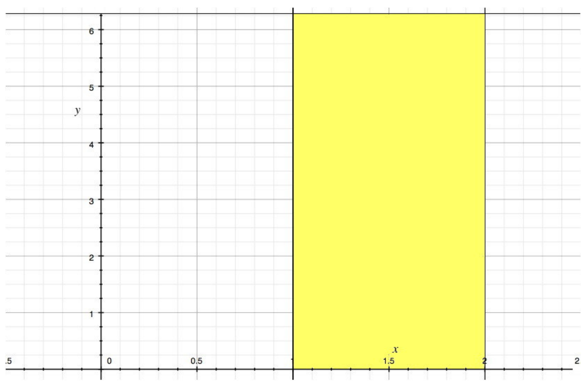
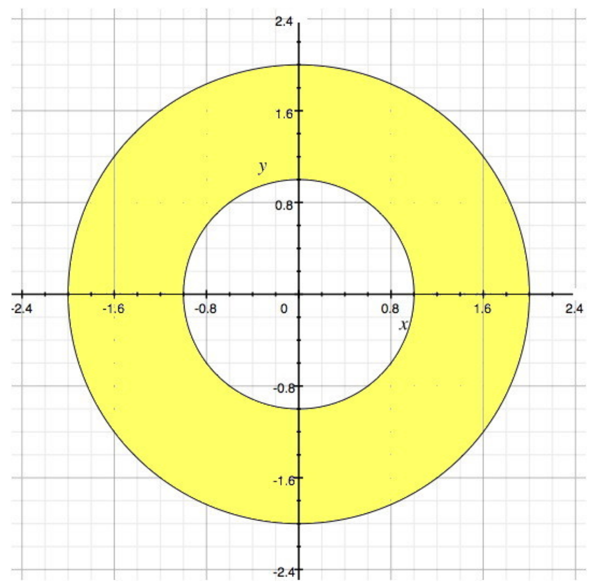
Remember $\displaystyle\iint_R f(x,y)\,dA$ is a number. The point of this section is to figure out how to figure out what this number is when it turns out that the region R can be easily described using polar coordinates.

Did you notice how Stewart slipped in vector functions of a vector variables? He called them "polar coordinates" but notice $\mathbf{P}\big(\langle r,\theta\rangle\big) = \langle r\cos\theta, r\sin\theta\rangle$ has domain 2-vectors and range also 2-vectors.



$\Rightarrow$



D $\qquad\qquad$ R

The formula is

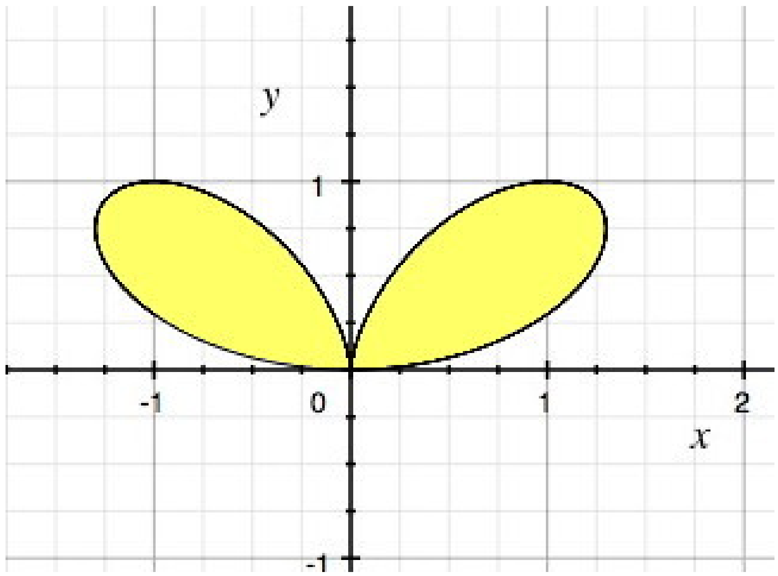
$$\iint_R f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,|r|\;dA$$

The absolute value is there because whatever is there is the ratio of two areas and hence positive.

There are two ways to approach the resulting iterated integral. If you see the yellow region in $\theta - r$ space then just set it up as an iterated integral from before since that what it is.

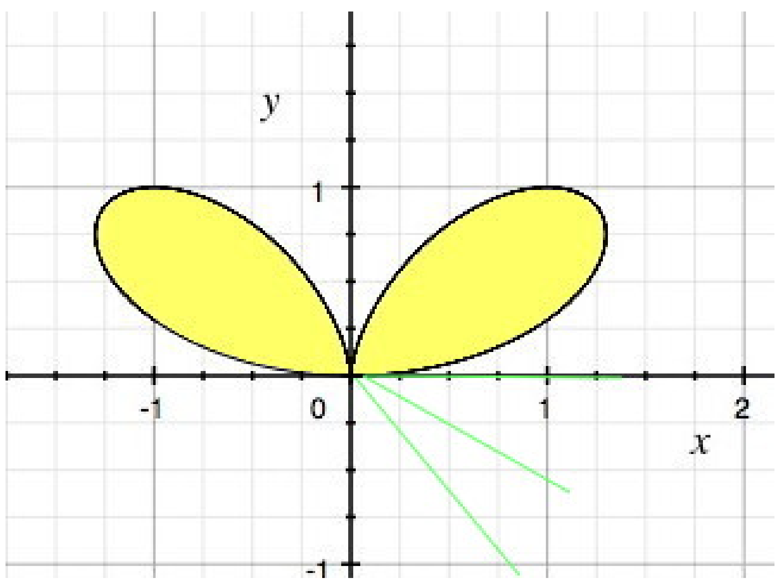
The second is to deal directly with the graph. Suppose we want to write the double integral as $dr\,d\theta$. This is by far the more common case.

Start with your region, say $r = g(\theta)$.

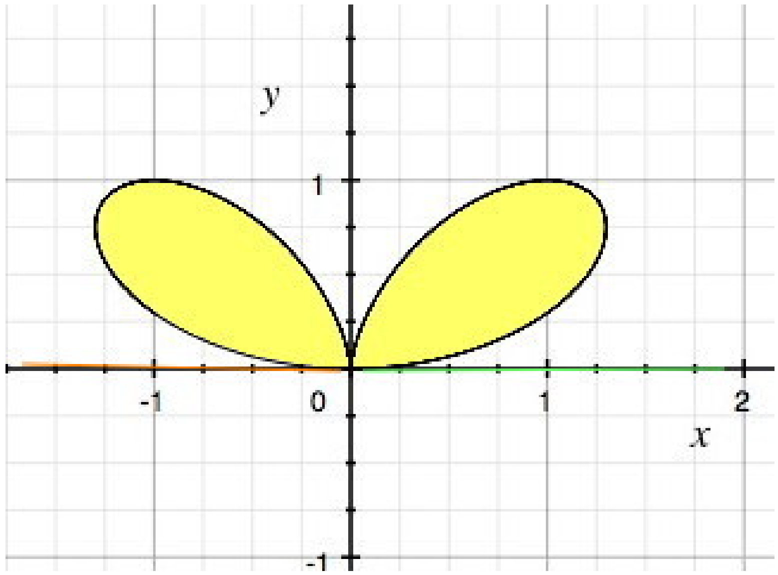


Since the outer variable is θ we want to start θ somewhere and keep going until we have swept out the region once (and only once).

Start with the lower limit:



Looks like $\theta = 0$ is where the swinging ray first hits the region, hence the lower limit is 0.

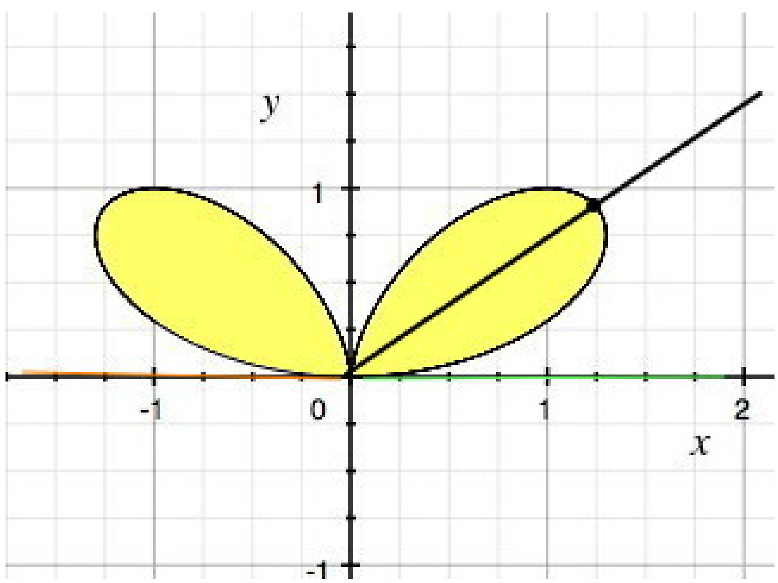


The upper limit looks like π to me. Hence

$$\iint_R \cdots \, dA = \int_0^{\pi} \int_?^? \cdots |r| \, dr \, d\theta$$

Polar graphs can be tricky. You may not have a correct interval but you can check this at the same time as you worry about the limits on the inner integral.

This time you have to actually see how the graph is drawn and follow the generic black ray. At $\theta = \dfrac{\pi}{4}$ we see

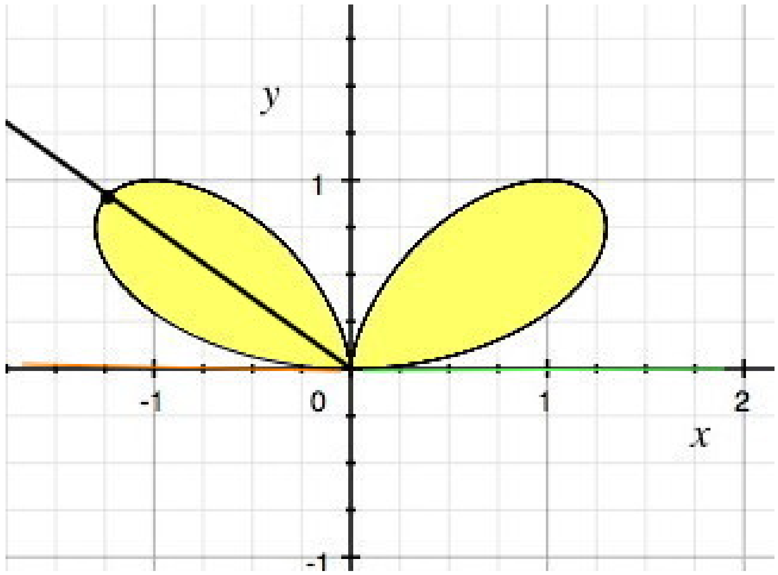


so thus far we are tracing out the curve. We need a formula for the value of r where the ray first enters the region. This appears to be $r = 0$. We need a formula for the value of r where the ray leaves enters the region. This appears to be $r = g(\theta)$. (And we should check that the entire ray between 0 and $g(\theta)$ lies in the region - it does.) Hence

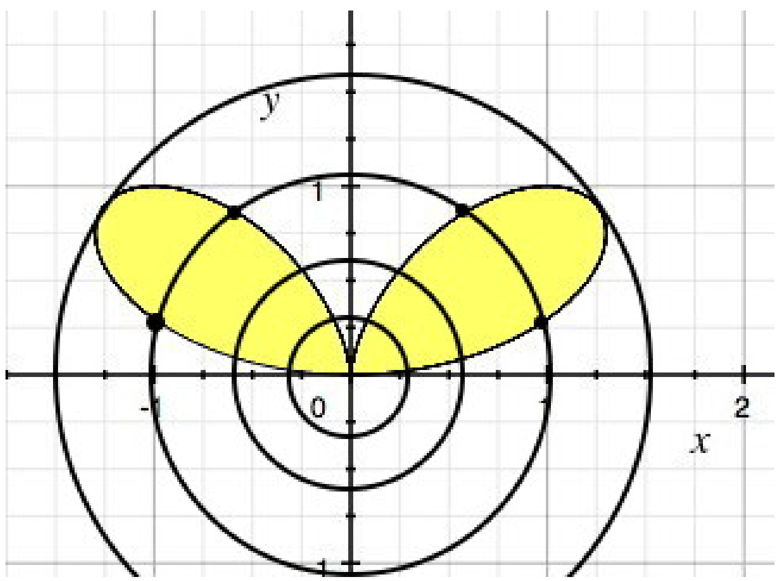
$$\iint_R \cdots \, dA = \int_0^{\pi} \int_0^{g(\theta)} \cdots \, r dr \, d\theta$$

and we have dropped the absolute value because r is positive. You should check a few more values of θ. At $\theta = \dfrac{\pi}{2}$ you have swept out the first ear, the ray starts at 0 and ends at $g(\theta) > 0$ and the whole interval lies in the region.

Here is the picture at $\dfrac{3\pi}{4}$:



The setup for $d\theta\, dr$ comes up much less in practice, but here is the relevant picture.

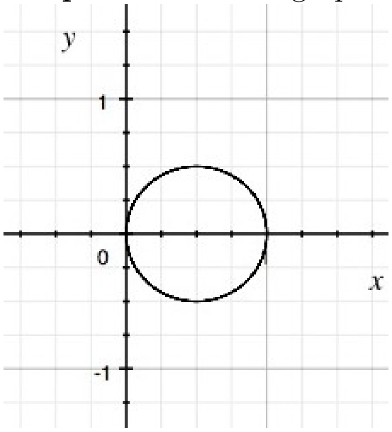


You look at expanding concentric circles and ask for what r do you first hit the region. Looks like $r = 0$ here. Then you keep going until the circles no longer hit the region: looks like about 1.5 to me.

$$\iint_R \cdots \, dA = \int_0^{1.5} \int_?^? \cdots \, r d\theta \, dr$$

The limits on the inner integral are functions of r and consist of the θ where the circle first enters the region and the θ where the circle leaves the region. In this picture that will have to be broken up into two integrals.

Example: Here is the graph of $r = \cos\theta$

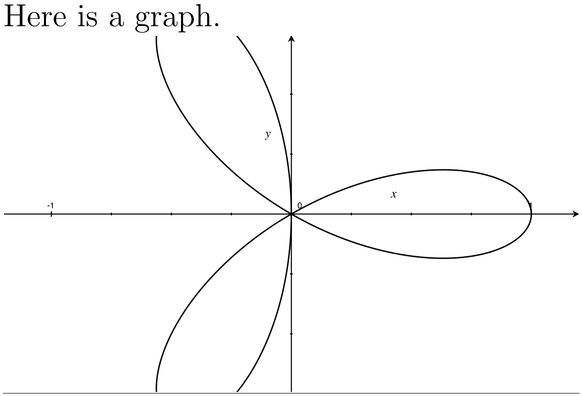


If you start with $\theta = 0$ and go once around the circle you get $\theta = \pi$. Hence the area of this circle is

$$\iint_C 1\, dA = \int_0^\pi \int_0^{\cos\theta} r\, dr\, d\theta = \int_0^\pi \frac{\cos^2\theta}{2} d\theta = \int_0^\pi \frac{1+\sin(2\theta)}{4} d\theta = \frac{\pi}{4} = \pi\left(\frac{1}{2}\right)^2$$

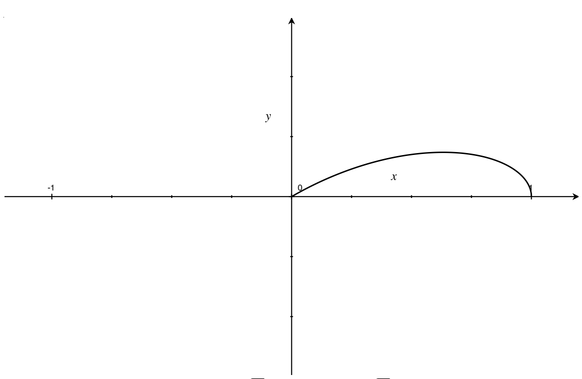
Example: Find the area enclosed by one leaf of the three leafed rose, $r = \cos(3\theta)$.

Here is a graph.

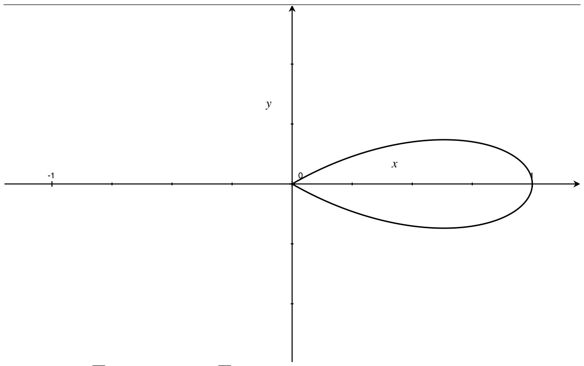


Since we have $r = \cos(3\theta)$, let's try to set it up as $dr\, d\theta$.

Just seeing the graph is not so helpful so lets start at $\theta = 0$, which is the point on the far right, $(1, 0)$ in both polar and rectangular coordinates. Now $\cos(\theta) \geqslant 0$ for $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$ so as θ increases we start traversing the top part of the right-hand pedal and we get to the origin at $\dfrac{\pi}{6}$ and we see



Hence is we let $-\dfrac{\pi}{6} \leqslant \theta \leqslant \dfrac{\pi}{6}$ we get



For $-\dfrac{\pi}{6} \leqslant \theta \leqslant \dfrac{\pi}{6}$, $r \geqslant 0$.

The area is $\displaystyle\iint_D 1\, dA$ so

$$\iint_D 1\, dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos(3\theta)} r\, dr\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^2(3\theta)}{2}\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\sin(6\theta)}{4}\, d\theta =$$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4}\, d\theta + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\sin(6\theta)}{4}\, d\theta = \frac{\pi}{12}$$

WARNING:

$\boxed{2}$ **Change to Polar Coordinates in a Double Integral** If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$\boxed{3}$ If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$