MATH 20550 Stokes' Theorem and the Divergence Theorem Fall 2016

These theorems loosely say that in certain situations you may replace one integral by a different one and get the same answer.

1. The Divergence Theorem

Let E be a bounded solid in three space and let ∂E be the boundary surface oriented so the unit normal points away from the surface (often called outward).

Let $\mathbf{F}$ be a field defined on an open set containing E such that the partials of $\mathbf{F}$ are continuous on E. Then

$$\iint\limits_{\partial E} \mathbf{F} \bullet d\mathbf{S} = \iiint\limits_{E} (\nabla \bullet \mathbf{F})\, dV$$

2. Stokes' Theorem

In Stokes' Theorem the situation is the following. You have an oriented surface with boundary, say T with boundary ∂T. Suppose $\mathbf{F}$ is a field defined on an open set containing T such that the partials of $\mathbf{F}$ are defined and continuous on that open set. Then, *with the correct orientations*,

$$\oint\limits_{\partial T} \mathbf{F} \bullet d\mathbf{r} = \iint\limits_{T} (\nabla \times \mathbf{F}) \bullet d\mathbf{S}$$

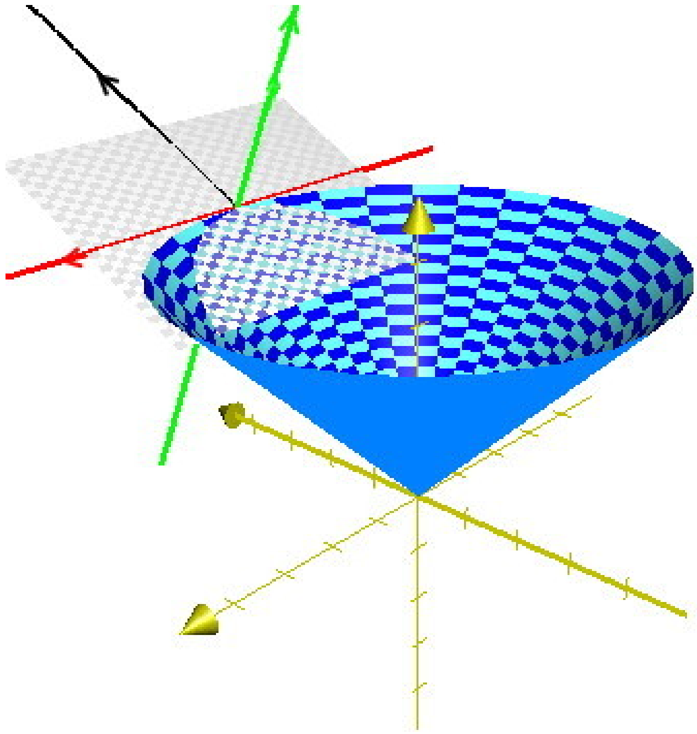
We will say that the orientation on the boundary curve and the orientation on the surface are *compatible* provided that, with these orientations, Stokes' Theorem holds.

Discussion. The surface integral is only defined once T is oriented and the line integral is only defined once ∂T is oriented. If different orientations are chosen, these numbers can change so in order for Stokes' Theorem to be correct, the normal vector to the surface and the tangent vector to the curve must be chosen compatibly. If both orientations in a compatible one are reversed then both sides are replaced by their negatives and so are still equal. Hence if one set of orientations is compatible, so is the set with both orientations reversed.

This of course requires that we have some way of determining whether two orientations are compatible or not. We do this as follows.

2.1. **Compatible orientations.** Given an oriented surface with boundary, pick a point on the boundary, say **p**. We have a normal vector to the surface at **p** and also a tangent plane at that point. In the tangent plane we have a tangent vector to the boundary curve and a normal vector in the tangent plane to the tangent line.

Here is a picture.



The gray plane is tangent to the surface at the point on the boundary where all three, mutually perpendicular, colored lines meet. The green line is normal to the tangent plane and hence is a normal line to the surface. The red line lies in the tangent plane and is the tangent line to the boundary curve. The black line lies in the tangent plane and is perpendicular to the tangent line.

For Stokes' Theorem, the green and red lines need to be compatibly oriented. The black line has a preferred orientation, namely go away from the surface. This is indicated in the picture by the black arrow. Neither the green nor the red line is naturally oriented but we can link the orientations via the right-hand rule as follows:

The red-green-black frame should be a right-hand frame.

In other words, if you put your right hand down with your fingers pointing in the red direction and curl them in the green direction, you must get the direction for the black line.

A better way to remember this since you usually won't have colored pictures is the following. If you put your fingers in the direction of the tangent to the boundary curve and curl them in the direction of the normal vector to the surface, your thumb should point out from the surface:

The tangent-normal-out should be a right-hand frame.

Another version of the same rule is to stand on the tangent plane with your head pointing in the green direction and face the red direction. Then the surface should be on your left (or the black line should point to your right).

Two orientations are *compatible* provided the "tangent-normal-out" frame is a right-hand frame at every point on the boundary curve.

By continuity, it is only necessary to check the compatibility condition at one point on each piece of your parametrization of the boundary.

3. General consequences

Both theorems say that one integral can be replaced by another and so there are problems where you are asked to do one and end up preferring to do the other. Examples will be provided below.

Obvious general results are

- Two fields with the same divergence over E have the same flux integrals over ∂E.
- Two fields with the same curl over T have the same line integral around ∂T.

Both theorems provide a proof of

$$\iint\limits_{\partial E} (\nabla \times F) \bullet d\mathbf{S} = 0$$

From the Divergence Theorem, $\iint\limits_{\partial E} (\nabla \times F) \bullet d\mathbf{S} = \iiint_E \nabla \bullet \big(\nabla \times F\big) dV = \iiint_E 0\, dV = 0.$

The proof via Stokes' Theorem is a bit more difficult. Divide the surface ∂E into two pieces T_1 and T_2 which meet along a common boundary curve. Then $\iint\limits_{\partial E} (\nabla \times F) \bullet d\mathbf{S} = \iint\limits_{T_1} (\nabla \times F) \bullet d\mathbf{S} + \iint\limits_{T_2} (\nabla \times F) \bullet d\mathbf{S}$. By Stokes' Theorem $\iint\limits_{T_i} (\nabla \times F) \bullet d\mathbf{S} = \oint\limits_{\partial T_i} \mathbf{F} \bullet d\mathbf{r}$. BUT $\partial T_1 = -\partial T_2$ so $\iint\limits_{T_1} (\nabla \times F) \bullet d\mathbf{S} + \iint\limits_{T_2} (\nabla \times F) \bullet d\mathbf{S} = 0$.

The proof via Stokes' Theorem is more complicated but it gives a better result because for the Divergence Theorem proof the partial derivatives of $\mathbf{F}$ need to be defined on a neighborhood of E whereas in the Stokes' Theorem proof the partial derivatives of $\mathbf{F}$ need only be defined on a neighborhood of ∂E.

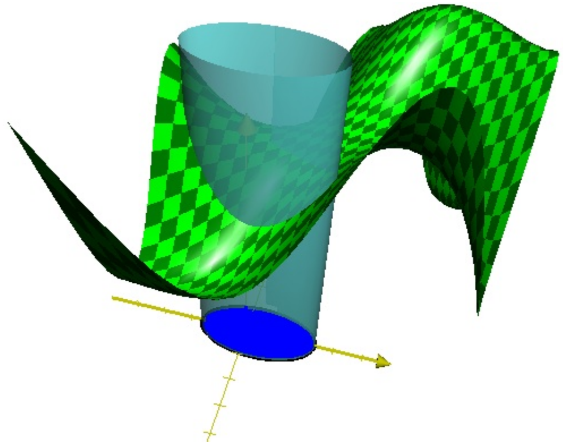
There are unexpected applications which sometimes lead to unexpected science and engineering results. The following was certainly unexpected (by me).

Let $z = f(x, y)$ be a graph of a differentiable function. Let D be a bounded region in the plane and let T be the part of the graph lying inside the cylinder along D. Then

$$\oint\limits_{\partial T} \langle 0, x, 0 \rangle \bullet d\mathbf{r} = \text{Area}(D)$$

In particular, the line integral is independent of the function f and doesn't change if you slide D around in the xy plane!

Here is one possibility for f and D. The green checkerboard is the graph of f. The solid blue ellipse in the xy plane is D and the curve ∂T is the intersection of the cylinder over ∂D and the graph of f.



Example

Once you think to look, the result is easy using Stokes' Theorem.

- $\nabla \times \langle 0, x, 0\rangle = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} =< 0, 0, 1 >$
- $d\mathbf{S} = \left\langle 1, 0, \dfrac{\partial f}{\partial x} \right\rangle \times \left\langle 0, 1, \dfrac{\partial f}{\partial y} \right\rangle = \left\langle -\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1 \right\rangle$

Hence $\displaystyle\iint_T (\nabla \times F) \bullet d\mathbf{S} = \iint_D 1 dA = \text{Area}(D)$

A more classical example is passage from integral equations to partial differential equations. A field, thought of as a flow field, is called *incompressible* provided $\displaystyle\iint_{\partial E} \mathbf{F} \bullet d\mathbf{S} = 0$ over all solids in the flow. A flow is called *irrotational* provided $\displaystyle\oint_{\partial T} \mathbf{F} \bullet d\mathbf{r} = 0$ over all surfaces with boundary in the flow. There are flows in physics and engineering which are incompressible (water is an often cited example) and others which are irrotational. The Wikipedia page for vorticity has an interesting discussion of irrotational flows.

- If a flow is incompressible, $\nabla \bullet \mathbf{F} = 0$.
- If a flow is irrotational, $\nabla \times \mathbf{F} = \langle 0, 0, 0\rangle$.

The second condition shows irrotational flows are gradient fields in simply-connected regions. Hence flows which are both incompressible and irrotational have potential functions which are harmonic.

The first condition shows incompressible flows are curls in 2-connected regions. We do not pursue this remark in this course since we have no good procedure for finding the field of which $\mathbf{F}$ is the curl, unlike the potential function case.

To see why the second condition holds, fix a point $\mathbf{p}$ in the flow and consider $\mathbf{c} = (\nabla \times \mathbf{F})(\mathbf{p})$. Suppose the z-component of $\mathbf{c}$, $\mathbf{c}_z$, is positive.

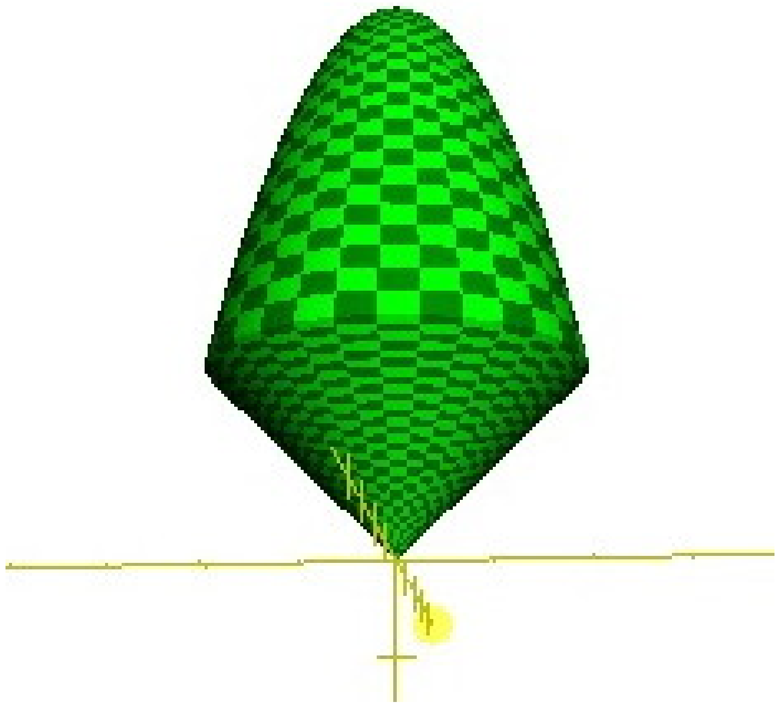
Then the z-component of $\nabla \times \mathbf{F}$ is positive in a little neighborhood of $\mathbf{p}$ by continuity. Let T be a ball in the plane $z = \mathbf{c}_z$ and use the upward normal on T. Then $\displaystyle\iint_T (\nabla \times \mathbf{F}) \bullet d\mathbf{S} > 0$ since $d\mathbf{S} = \langle 0, 0, 1\rangle\, dA$ on T. But by Stokes' Theorem $\displaystyle\iint_T (\nabla \times \mathbf{F}) \bullet d\mathbf{S} = \oint_{\partial T} \mathbf{F} \bullet dr = 0$. Hence $\mathbf{c}_z$ can not be positive. A similar argument shows it can not be negative so $\mathbf{c}_z = 0$.

Similar arguments show the other two coordinates are 0 so $\nabla \times \mathbf{F} = \langle 0, 0, 0\rangle$.

The proof of the first result uses the Divergence Theorem together with the same idea to show that the function $\nabla \bullet \mathbf{F}$ at a point can not be positive nor can it be negative.

4. Examples

Example: Integrate the field $\mathbf{H} = \left\langle xy, -\dfrac{y^2}{2}, z \right\rangle$ over the surface which is a paraboloid of revolution, $z = 6 - x^2 - y^2$ for $x^2 + y^2 \leqslant 4$ and the cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leqslant 4$.



We could write the surface as $T_1 \cup T_2$ where T_1 is the paraboloid and T_2 is the cone, do the two surface integrals and add the results (being careful with orientations). Or, we could use the Divergence Theorem.

$$\iint\limits_{\partial E} \mathbf{H} \bullet d\mathbf{S} = \iiint\limits_{E} \nabla \bullet \mathbf{H}\, dV$$

In this case, $\nabla \bullet \mathbf{H} = y - y + 1 = 1$ so if we can set up the triple integral, this is the way to go.

Use cylindrical coordinates. The plane region is the disk of radius 2 or $0 \leqslant r \leqslant$. At a point (r, θ), $r \leqslant z \leqslant 6 - r^2$ so

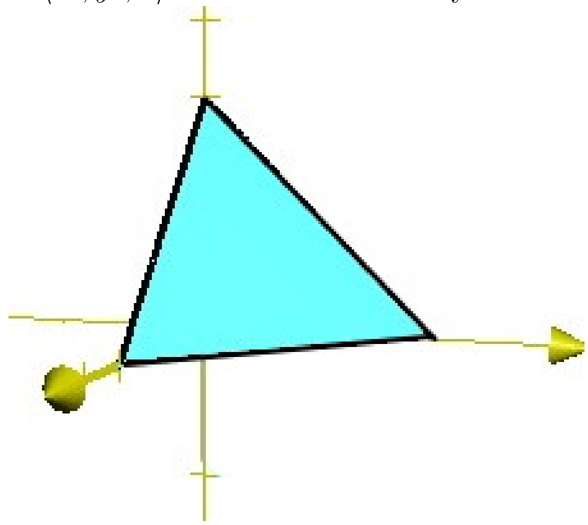
$$\iiint\limits_{E} \nabla \bullet \mathbf{H}\, dV = \iint\limits_{D} \int\limits_{r}^{6-r^2} dz\, dA = \int\limits_{0}^{2\pi} \int\limits_{0}^{2} \int\limits_{r}^{6-r^2} r\, dz\, dr\, \theta =$$

$$\int\limits_{0}^{2\pi} \int\limits_{0}^{2} r\, z\Big|_{r}^{6-r^2}\, dr\, \theta = \int\limits_{0}^{2\pi} \int\limits_{0}^{2} r(6 - r - r^2)\, dr\, \theta = \int\limits_{0}^{2\pi} \int\limits_{0}^{2} (6r - r^2 - r^3)\, dr\, \theta =$$

$$\int\limits_{0}^{2\pi} 3r^2 - \frac{r^3}{3} - \frac{r^4}{4}\Big|_{0}^{2} d\theta = \int\limits_{0}^{2\pi} 12 - \frac{8}{3} - 4 d\theta = 2\pi \frac{16}{3} = \frac{32\pi}{3}$$

Warning! We do not know if this is the right answer, nor can we know since we were not told which normal to the surface to use. Get in the habit of wondering what the normal vectors (and tangent vectors for Stokes' Theorem) are!

Example: Let T be the triangle in the plane $x+y+z=1$ with vertices $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$ and $\langle 0,0,1\rangle$. Integrate the field $\mathbf{G}=\langle z^2,y^2,x\rangle$ around the boundary of T counterclockwise.



We can do this by parametrizing the three line segments, doing three line integrals and combining the results, being careful with orientations.

Or we can try Stokes' Theorem. First compute $\nabla\times G = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = \langle 0, 2z-1, 0\rangle$.

Parametrize the blue T by $\langle x,y,1-x-y\rangle$ with $(x,y)\in D$ where D is the triangle in the first quadrant of the xy plane with vertices $\langle 1,0\rangle$ and $\langle 0,1\rangle$. The normal is $\left\langle -\dfrac{z}{x}, -\dfrac{z}{y}, 1\right\rangle = \langle 1,1,1\rangle$ so

$$\oint\limits_{\partial T} \mathbf{G}\bullet d\mathbf{r} = \iint\limits_{D} \langle 0,2z-1,0\rangle \bullet \langle 1,1,1\rangle\; dA$$

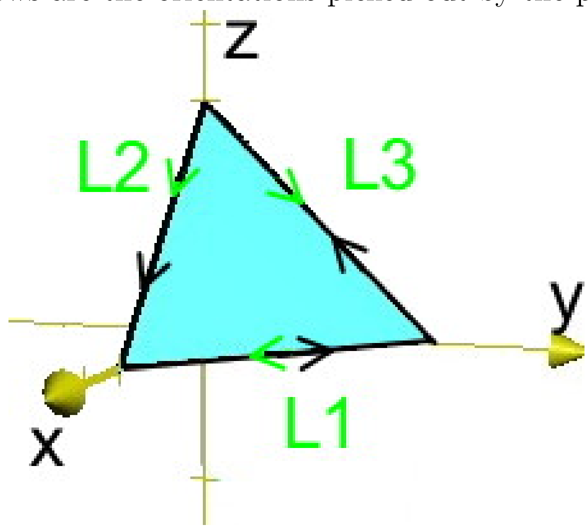
$$\iint\limits_{D} 2z-1\,dA = \int_0^1\int_0^{1-x} 2(1-x-y)-1\,dy\,dx = \int_0^1\int_0^{1-x} 1-2x-2y\,dy\,dx = \int_0^1 y-2xy-y^2\Big|_0^{1-x} dx =$$

$$\int_0^1 \left((1-x)-2x(1-x)-(1-x)^2\right)dx = \int_0^1 x^2-x\;\;x = \frac{x^3}{3}-\frac{x^2}{2}\Big|_0^1 = -\frac{1}{6}$$

Here is what is involved in computing the line integral directly.
Step 1: Parametrize the boundary.
$L_1\colon \mathbf{r}_1(t)=\langle t,1-t,0\rangle$,
$L_2\colon \mathbf{r}_2(t)=\langle t,0,1-t\rangle$ and
$L_3\colon \mathbf{r}_3(t)=\langle 0,t,1-t\rangle$ where $0\leqslant t\leqslant 1$ in all three cases.
Step 2: Compute tangent vectors.
$L_1\colon \mathbf{r}_1'(t)=\langle 1,-1,0\rangle$,
$L_2\colon \mathbf{r}_2'(t)=\langle 1,0,-1\rangle$ and
$L_3\colon \mathbf{r}_3'(t)=\langle 0,1,-1\rangle$.
Step 3: Set up and do the integrals.
$\int\limits_{L_1} \mathbf{G}\bullet d\mathbf{r} = \int_0^1 \langle 0,(1-t)^2,t\rangle \bullet \langle 1,-1,0\rangle\; dt = \int_0^1 -(1-2t+t^2)\,dt = -t+t^2-\frac{t^3}{3}\Big|_0^1 = -\frac{1}{3}$,

$\int\limits_{L_2} \mathbf{G} \bullet d\mathbf{r} = \int\limits_0^1 \langle (1-t)^2, 0, t\rangle \bullet \langle 1, 0, -1\rangle \; dt = \int\limits_0^1 (1-t)^2 - t\, dt = \int\limits_0^1 1 - 3t + t^2\, dt = t - \frac{3t^2}{2} + \frac{t^3}{3}\Big|_0^1 = -\frac{1}{6}$
and
$\int\limits_{L_3} \mathbf{G} \bullet d\mathbf{r} = \int\limits_0^1 \langle (1-t)^2, t^2, 0\rangle \bullet \langle 0, 1, -1\rangle \; dt = \int\limits_0^1 t^2\, dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3},$

Next is a picture of the triangle again. The black arrows indicate the requested orientation (clockwise) and the green arrows are the orientations picked out by the parametrization.



Hence

$$\oint\limits_{\partial T} \mathbf{G} \bullet d\mathbf{r} = -\int\limits_{L1} \mathbf{G} \bullet d\mathbf{r} + \int\limits_{L2} \mathbf{G} \bullet d\mathbf{r} - \int\limits_{L3} \mathbf{G} \bullet d\mathbf{r} = -\left(-\frac{1}{3}\right) + \left(-\frac{1}{6}\right) - \frac{1}{3} = -\frac{1}{6}$$

Example: We did this example in the last class. Let T the unit cube in the first octant. Orient the surface so that the normal is outward from the solid cube. Integrate the field $\mathbf{F} = \langle x, y, z\rangle$ over this surface.

Using the Divergence Theorem, $\nabla \bullet \mathbf{F} = \dfrac{\partial\, x}{\partial\, x} + \dfrac{\partial\, y}{\partial\, y} + \dfrac{\partial\, z}{\partial\, z} = 1 + 1 + 1 = 3.$

$$\iint\limits_T \mathbf{F} \bullet \, d\mathbf{S} = \iiint\limits_E (\nabla \bullet \mathbf{F})\, dV = 3 \cdot \text{volume of unit cube} = 3$$

5. Further remarks

Typically applications of these theorems goes from the integral over the boundary to an integral over the whole thing since otherwise you have to find a field whose curl or divergence is the field you want to integrate.

Here in one place are the three Stokes'-type theorems.

(1) $p(\mathbf{b}) - p(\mathbf{a}) = \displaystyle\int_C \nabla p \bullet d\mathbf{r}$

(2) $\displaystyle\oint_{\partial T} \mathbf{F} \bullet d\mathbf{r} = \iint_T (\nabla \times \mathbf{F}) \bullet d\mathbf{S}$

(3) $\displaystyle\iint_{\partial E} \mathbf{F} \bullet d\mathbf{S} = \iiint_E (\nabla \bullet \mathbf{F})\, dV$

Green's Theorem is missing from the list because it is just Stokes' Theorem with $\mathbf{F} = \langle M(x,y), N(x,y), 0\rangle$ and T is parametrized by $\langle x, y, 0\rangle$ with $(x,y) \in T$.