1. Definition

Given a solid S in space, a *partition of* S consists of a finite set of solis $\mathcal{S} = \{S_1, \cdots, S_n\}$ such that the S_i cover S, or equivalently $S \subset \bigcup\limits_{i=1}^{n} S_i$. Furthermore, for each S_i, S intersects S_i or $S_i \cap S \neq \emptyset$. Finally $S_i \cap S_j$ for $i \neq j$ is either empty or a surface. A *mesh of* $\mathcal{S}$ is a number m such that each S_i is contained in a ball of radius m (the center of this ball can change for each S_i).

Next pick a point $p_i = (x_i, y_i, z_i)$ in each $S_i \cap S$. Then the Riemann sum associated to the partition $\mathcal{S}$, the points p_i and the function $f(x, y, z)$ is

$$\mathrm{RS}(\mathcal{S}, \{p_i\}, f) = \sum_{i=1}^{n} f(x_i, y_i, z_i) \cdot \mathrm{volume}(S_i)$$

$$\iiint_S f(x, y, z)\, dV = \lim_{\mathrm{mesh} \to 0} \mathrm{RS}(\mathcal{S}, \{p_i\}, f)$$

provided this limit exists.

A basic result due to Riemann is the following

If f is continuous on S and if S is closed and bounded, then $\iiint_S f(x, y, z)\, dV$ exists.

Other basic results which follow from the definition.

If f and g satisfy $f \leqslant g$ on S and if $\iiint_S f(x, y, z)\, dV$ and $\iiint_S g(x, y, z)\, dV$ exist, then

$$\iiint_S f(x, y, z)\, dV \leqslant \iiint_S g(x, y, z)\, dV$$

The integrals are equal if and only if the functions are equal.

A corollary of this result is that if $m \leqslant f(x, y, z) \leqslant M$ on R then

$$m \cdot \mathrm{volume}(S) \leqslant \iiint_S f(x, y, z)\, dV \leqslant M \cdot \mathrm{volume}(S)$$

provided $\iiint_S f(x, y, z)\, dV$ exists.

If f and g are defined on S and if $\iiint_S f(x, y, z)\, dV$ and $\iiint_S g(x, y, z)\, dV$ exist, then

$$\iiint_S f(x, y, z) + g(x, y, z)\, dV = \iiint_S f(x, y, z)\, dV + \iiint_S g(x, y, z)\, dV$$

If f is defined on S, if c is a constant, and if $\iiint_S f(x, y, z)\, dV$ exists, then

$$\iiint_S c \cdot f(x, y, z)\, dV = c \iiint_S f(x, y, z)\, dV$$

If $S = S_1 \cup S_2$ and if $S_1 \cap S_2$ is contained in a surface, then

$$\iiint_S f(x, y, z)\, dV = \iiint_{S_1} f(x, y, z)\, dV + \iiint_{S_2} f(x, y, z)\, dV$$

provided $\iiint_{S_1} f(x,y,z)\,dV$ *and* $\iiint_{S_2} f(x,y,z)\,dV$ *exist.*

Midpoint Rule: If S is partitioned into boxes $x_0 < x_1 < \cdots < x_n$, $y_0 < y_1 < \cdots < y_m$ and $z_0 < z_1 < \cdots < z_r$ and then

$$\iiint_S f(x,y,z)\,dV \approx \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} f(\bar{x}_i, \bar{y}_j, \bar{z}_k) \cdot |x_i - x_{i-1}| \cdot |y_j - y_{j-1}| \cdot |z_j - z_{j-1}|$$

where $\bar{x}_i = \dfrac{x_i + x_{i-1}}{2}$, $\bar{y}_j = \dfrac{y_j + y_{j-1}}{2}$ and $\bar{z}_k = \dfrac{z_k + z_{k-1}}{2}$.

Average Value By definition, the average value of a function f on a solid S is

$$\frac{1}{\text{volume}(S)} \iiint_S f(x,y,z)\,dV$$

2. Iterated integrals

An iterated (triple) integral is an expression of the form

$$\int_a^b \int_{b(x)}^{t(x)} \int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz\,dy\,dx$$

There will be other variations. Six of them can be obtained by permuting the order of the variables. Others come from using different coordinate systems.

It is also useful in setting up triple integrals as iterated integrals to let R be the region defined by $\displaystyle\int_a^b \int_{b(x)}^{t(x)} \cdots\,dy\,dx$ and observe that

$$\int_a^b \int_{b(x)}^{t(x)} \int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz\,dy\,dx = \iint_R \left(\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz \right) dA$$

The *outer double integral* is an ordinary double integral so if you have $\displaystyle\iint_R \left(\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz \right) dA$ you know how to get the corresponding iterated integral. The new material is to work out what

$$\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz$$

is and you can certainly guess the correct answer. Here $a(x,y)$ and $b(x,y)$ are functions of x and y where usually $a(x,y) \leqslant b(x,y)$ for $(x,y) \in R$ but the inequality is not necessary to work out the answer so don't bother checking it at this point. All you do is do a first year calculus definite integral treating x an y as constants.

In more detail, by the Fundamental Theorem of Calculus, you need to find $F(x,y,z)$ such that

$$\frac{\partial F}{\partial z}(x,y,z) = f(x,y,z)$$

and then

$$\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz = F\big(x,y,b(x,y)\big) - F\big(x,y,a(x,y)\big)$$

Notice as promised that $\displaystyle\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz$ is a function of x and y so the outer double integral is just an ordinary double integral.

3. Setting up iterated integrals

The goal is to reduce a triple integral to an iterated integral. As is usual in this sort of problem, the function is irrelevant. Start with a solid S and pick a coordinate plane, say the xy plane. You are looking for the region R in the xy plane with the following property. A point $(x, y) \in R$ if and only if there is an interval $[\alpha, \omega]$ such that the vertical line through the point (x, y), say $\langle x, y, 0\rangle + t\,\langle 0, 0, 1\rangle$ intersects S in the segment $\langle x, y, 0\rangle + t\,\langle 0, 0, 1\rangle$, $\alpha \leqslant t \leqslant \omega$. In particular, if $(x, y) \notin R$, the line $\langle x, y, 0\rangle + t\,\langle 0, 0, 1\rangle$ does not intersect the solid S at all.

For each value of (x, y) in R, the point $\langle x, y, \alpha\rangle$ is the lowest point in S which intersects the line. Hence α is a function of (x, y) and we write $\alpha(x, y)$. Similarly, $\langle x, y, \omega\rangle$ is the highest point in S which intersects the line so $\omega = \omega(x, y)$. To proceed YOU need to find formulas for $\alpha(x, y)$ and $\omega(x, y)$ and then

$$\iiint_S f(x, y, z)\, dV = \iint_R \left(\int_{\alpha(x,y)}^{\omega(x,y)} f(x, y, z)\, dz \right) dA$$

The region R is called the *projection of the solid S into the xy plane.*

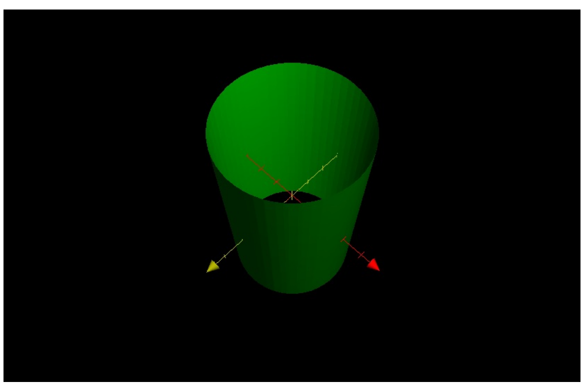
Hence given a solid S you need to determine:

(1) The projection of S into a coordinate plane (your choice).
(2) The functions α and ω.

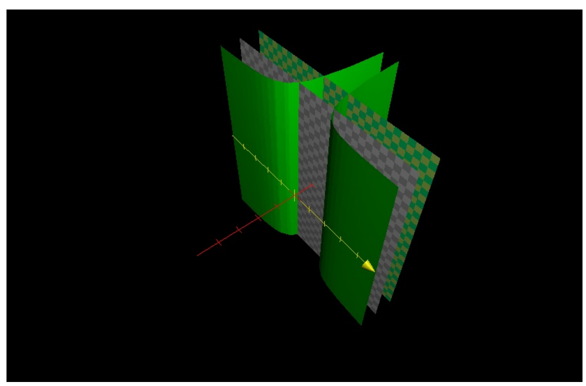
3.1. Projections.

As we discussed earlier in the semester, projections of cylinders are easy. A cylinder is a curve in a coordinate plane and consists of the set of all lines perpendicular to the coordinate plane passing through the curve.

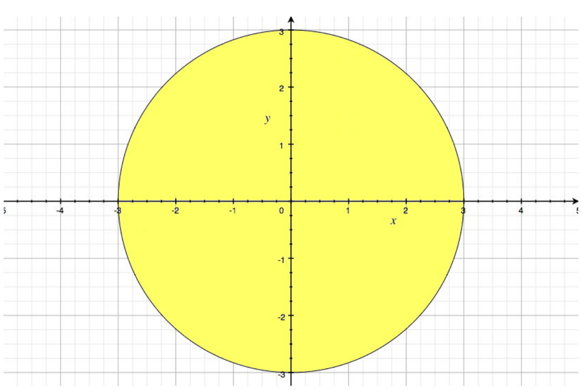
Examples of cylinders:

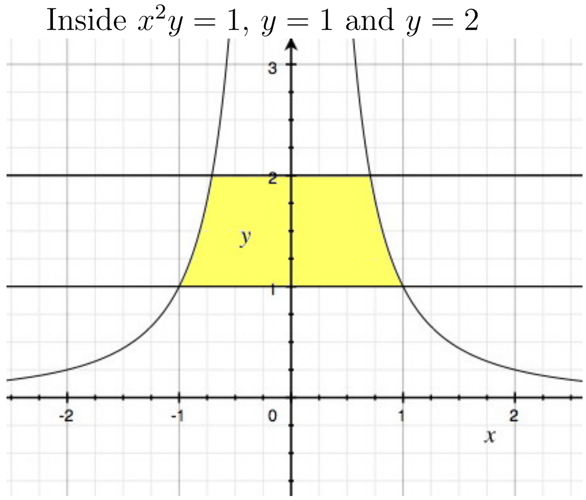


Inside $x^2 + y^2 = 9$

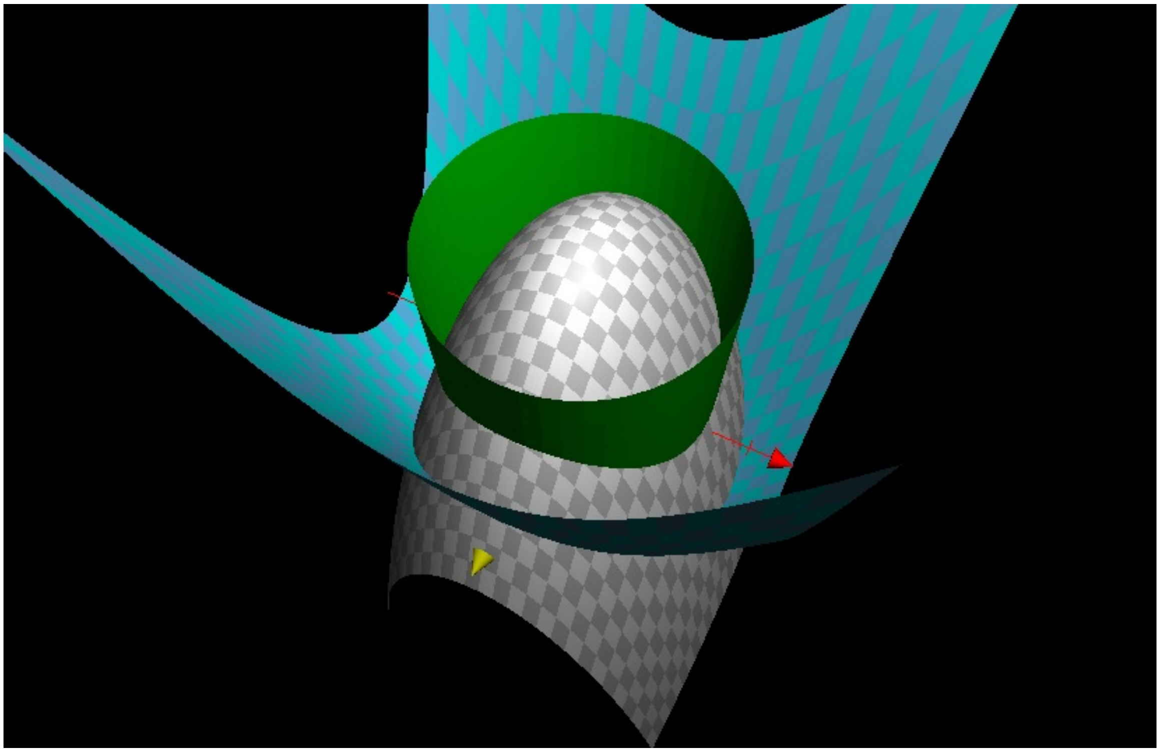


Inside $x^2y = 1$, $y = 1$ and $y = 2$





3.2. **Equations.** The solid S we are going to discuss has projection R where R is one of the two regions in the last subsection and it is the part of 3-space above $z = 20 - x^2 - 2y^2$ and below $z = 2x^2 + y^2 - 28$.



Here is a picture of the case of where R is the disk. The blue surface is the graph of $z = 2x^2+y^2-28$ while the white one is $z = 20 - x^2 - 2y^2$.

If $x^2 + y^2 \leqslant 9$, $-3 \leqslant x \leqslant 3$ and $-3 \leqslant y \leqslant 3$. Hence $x^2 + 2y^2 \leqslant 18$ and $2x^2 + y^2 \leqslant 18$ so inside the yellow disk, $z = 2x^2 + y^2 - 28$ is negative and $z = 20 - x^2 - 2y^2$ is positive so at any point (x, y) in R, the interval $[2x^2 + y^2 - 28, 20 - x^2 - 2y^2]$ lies in S.

Hence all of the yellow disk lies between these two graphs and

$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{2x^2+y^2-28}^{20-x^2-2y^2} f(x,y,z)\,dz \right) dA$$

To work out an example to the end, suppose we want to find the volume. Then $f(x, y, z) = 1$.

$$\iiint_S 1\,dV = \iint_R \left(\int_{2x^2+y^2-28}^{20-x^2-2y^2} 1\,dz \right) dA = \iint_R z\Big|_{z=2x^2+y^2-28}^{z=20-x^2-2y^2} dA =$$
$$\iint_R 3(16 - x^2 - y^2)\,dA = 3\int_0^{2\pi}\int_0^4 (16 - r^2) r\,dr\,d\theta = 6\pi \left(8r^2 - \frac{r^4}{4} \right)\Big|_0^4 =$$
$$6\pi\,(128 - 64) = 6 \cdot 64\pi = 384\pi$$

For another example, consider the surface T above $z = 2x^2 + y^2 - 28$ and below $z = 20 - x^2 - 2y^2$. At first there seems to be no obvious region. Each graph projects into the entire xy plane so these projections are not the issue.

However, the solid is the set of all points $2x^2 + y^2 - 28 \leqslant 20 - x^2 - 2y^2$. To understand this inequality, consider the equality $2x^2 + y^2 - 28 = 20 - x^2 - 2y^2$, or $x^2 + y^2 = 16$.

The circle divides the plane into two regions and inside each region either $2x^2 + y^2 - 28 \leqslant 20 - x^2 - 2y^2$ or $2x^2 + y^2 - 28 \geqslant 20 - x^2 - 2y^2$ by continuity.

At the origin $(0, 0)$, $2x^2 + y^2 - 28 < 20 - x^2 - 2y^2$ so the disk satisfies $2x^2 + y^2 - 28 \leqslant 20 - x^2 - 2y^2$. A point in the outside region is $(5, 0)$ and therefore $2x^2 + y^2 - 28 \geqslant 20 - x^2 - 2y^2$ in the entire outside region. Hence the disk $x^2 + y^2 \leqslant 16$ is the set of all (x, y) such that $2x^2 + y^2 - 28 \leqslant 20 - x^2 - 2y^2$. Then

$$\iiint_T f(x, y, z)\, dV = \iint_{x^2+y^2\leqslant 16} \left(\int_{2x^2+y^2-28}^{20-x^2-2y^2} f(x, y, z)\, dz \right) dA$$

4. Type ? solids

We will evaluate the triple integral

$$\iiint_E f(x, y, z)\, dV$$

where E is the tetrahedron below the plane $x + 2y + 3z = 12$ and in the first octant. We will do this in three ways.

The definition in the book of Type ? solids are simply those for which the projection into a coordinate plane can be worked out. In particular, we will see that the solid E is Type 1, Type 2 and Type 3.

If we project E into the xy plane we need to find all $x \geqslant 0$, $y \geqslant 0$ such that $z = \dfrac{12 - x - 2y}{3} \geqslant 0$. The line $x + 2y = 12$ divides the first quadrant into two pieces. The point $(0, 0)$ lies in the triangle and $z > 0$ there so the triangle is the projection. Hence the projection into the xy plane is the triangle in the first quadrant $x + 2y = 12$, denoted T_1.

If we project E into the xz plane we need to find all $x \geqslant 0$, $z \geqslant 0$ such that $y = \dfrac{12 - x - 3z}{2} \geqslant 0$. The line $x + 3z = 12$ divides the first quadrant into two pieces. The point $(0, 0)$ lies in the triangle and $y > 0$ there so, the triangle is the projection. Hence the projection into the xz plane is the triangle in the first quadrant $x + 3z = 12$, denoted T_2.

If we project E into the yz plane we need to find all $y \geqslant 0$, $z \geqslant 0$ such that $x = 12 - 2y - 3z \geqslant 0$. The line $2y + 3z = 12$ divides the first quadrant into two pieces. The point $(0, 0)$ lies in the triangle and $x > 0$ there so, the triangle is the projection. Hence the projection into the yz plane is the triangle in the first quadrant $2y + 3z = 12$, denoted T_3.

Then

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \iint_{T_1} \left(\int_0^{\frac{12-x-2y}{3}} f(x, y, z)\, dz \right) dA \\
&= \iint_{T_2} \left(\int_0^{\frac{12-x-3z}{2}} f(x, y, z)\, dy \right) dA \\
&= \iint_{T_3} \left(\int_0^{12-2y-3z} f(x, y, z)\, dx \right) dA
\end{aligned}$$

Notice that this gives triple iterated integral problems where the integral may be hard to do in one setup but easier in another.

5. From iterated to triple

Just as in the 2-dimensional case, given an iterated integral it is pretty easy to work out a description of the solid. Start with

$$\int_a^b \int_{a(x)}^{b(x)} \int_{\alpha(x,z)}^{\omega(x,z)} f(x,y,z)dy\,dz\,dx$$

and describe the solid S over which we are integrating. We have projected into the xz plane and there we have the region R above $z = a(x)$, below $z = b(x)$ and $a \leqslant x \leqslant b$. Over the region R in the xz plane we are above the graph $y = \alpha(x,z)$ and below the graph $y = \omega(x,z)$. Said another way

$$S = \Big\{(x,y,z) \;\Big|\; a \leqslant x \leqslant b, a(x) \leqslant z \leqslant b(x), \alpha(x,z) \leqslant y \leqslant \omega(x,z)\Big\}$$

Of course if you are ever lucky enough to get a solid described to you by three such inequalities, the setup of one iterated integral is immediate.

Warning: You may choose a, b, $a(x)$, $b(x)$, $\alpha(x,z)$ and $\omega(x,z)$ arbitrarily and the iterated integral makes sense if the functions are continuous. However, the inequalities above may not describe a solid. Built into the notation are the assumptions that $a < b$; that for any $x \in [a,b]$, $a(x) \leqslant b(x)$; and for any (x,z) with $x \in [a,b]$ and $a(x) \leqslant z \leqslant b(x)$, $\alpha(x,z) \leqslant \omega(x,z)$.

6. A philosophically satisfying alternate approach

One can also set up an iterated integral as follows. As usual, let S be our solid and this time, pick an axis, say z. Since S is bounded, there is a number a such that the plane $z = a$ just touches the solid from underneath. There is also a b such that the plane $z = b$ just touches the solid from above. For an arbitrary z between a and b, the plane at that height intersects the solid in a region $R(z)$ (the region changes as you change z). Then

$$\iiint_S f(x,y,z)\,dV = \int_a^b \iint_{R(z)} f(x,y,z)\,dA\,dz$$

This method is less common because for some reason people are not so fond of varying regions.

It can be useful in solving the rewriting problem as in Example 4 from the book (page 1045). Note that even with the 7th edition there are still typos. (See the coordinates of the regions.) The part that I don't like about the example is that the book just asserts the projections are what they are without much help to the reader. But figuring out the projections is the hard part!

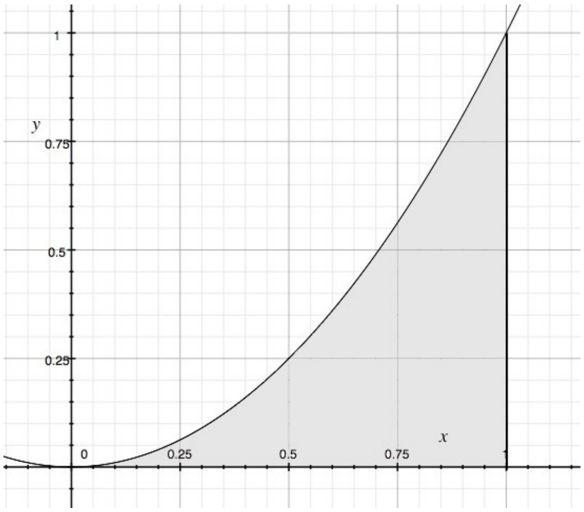
Anyway, we start with

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$$

To switch the outer two variables is easy (or at least it is a problem you have already studied).

$$\iiint_E f(x,y,z)\,dV = \iint_R \int_0^y f(x,y,z)\,dz\,dA$$

where R is



This is easy to set up the other way:

$$\iiint_E f(x,y,z)\,dV = \iint_R \int_0^y f(x,y,z)\,dz\,dA = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy$$

A harder problem is to switch the two inner variables, but here the alternate approach to setup is useful. Let us switch x and z in this last integral.

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1 \left(\iint_{R(y)} f(x,y,z)\,dA \right) dy$$

Here $R(y)$ is the rectangle in the xz plane where x runs between $\sqrt{y}$ and 1 and z runs between 0 and y. Since this is a rectangle, it is easy to switch the order of integration and get

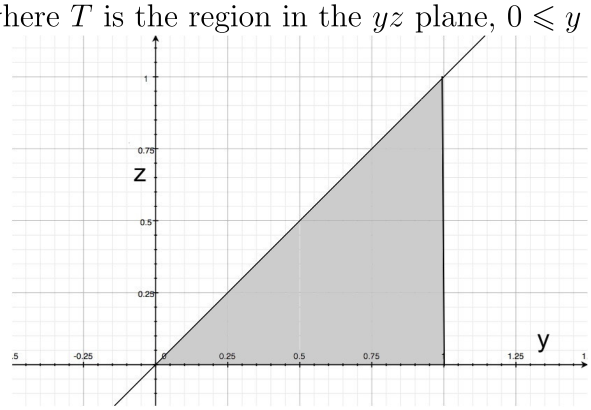
$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

which is the answer in the book.

If you now want to switch z and y in this last integral, this is the easy version.

$$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy = \iint_T \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dA$$

where T is the region in the yz plane, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant y$.



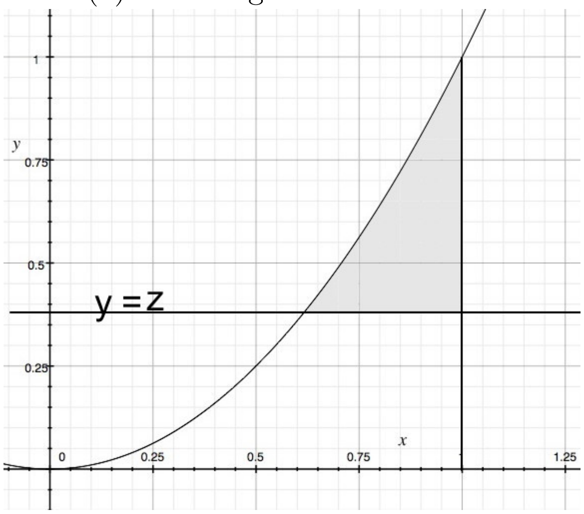
Hence

$$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy = \iint_T \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dA = \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz$$

To switch x and y in this last integral, write

$$\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \iint_{R(z)} f(x,y,z)\,dA\,dz$$

where $R(z)$ is the region



Reversing the order on $R(z)$ gives

$$\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \iint_{R(z)} f(x,y,z)\,dA\,dz = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz$$