12 Chapter 12

12.1 3-dimensional Coordinate System

The 3-dimensional coordinate system we use are coordinates on $\mathbb{R}^3$. The coordinate is presented as a triple of numbers: (a, b, c). In the Cartesian coordinate system we have an origin $(0,0,0)$, and three axis: the x-, y-, z-axes. These 3 axes are perpendicular to each other and their positive directions satisfy the "right hand rule": point your index finger on your right hand along the x-axis, curl it toward the y-axis, then your "thumb up" will point along the z-axis. Examples of properly drawn axes are:

(arrows denote positive direction)

To locate the point P which has coordinates (a, b, c): move a units in the x-direction, b in the y-direction, and c in the z-direction.
Ex. Plot $(2,1,3)$:

What would the equation $z = 3$ represent in $\mathbb{R}^3$?

The only restriction here is that $z = 3$, so any point of the form $(x, y, 3)$ satisfies this. This is a plane, parallel to the xy-plane, at "height" = 3:

How about $y = x^2$?

In the xy-plane, this is just a parabola, but in $\mathbb{R}^3$, this equation gives us no restriction on z, so the graph of the equation is

The coordinate planes are the xy-, xz-, and yz-planes, which are represented by $z = 0$, $y = 0$, and $x = 0$ respectively. Graphically:

We can also talk about "projecting" onto the coordinate planes. This is done by setting the appropriate coordinate to 0.

The projection of (a, b, c) onto the:

- xy-plane is $(a, b, 0)$
- xz-plane is $(a, 0, c)$

- yz-plane is $(0, b, c)$

Just as in the plane, we can talk about the distance between points. Applying the Pythagorean Theorem twice, we arrive at the distance formula.

Formula 1 (Distance Formula in $\mathbb{R}^3$)**.** *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. The distance from P_1 to P_2 is*

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(Note the book uses $|P_1P_2|$ instead of $d(P_1, P_2)$.)

Consider the point (h, k, l). Suppose we want an equation for the collection of points which are distance r away from (h, k, l). Using the distance formula, we know any point (x, y, z) satisfying this criteria satisfies:

$$r = d((h, k, l), (x, y, z)) = \sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2}$$

This set of points is the sphere with radius r and center (h, k, l). Squaring both sides of the equation, we arrive at a more friendly equation for the sphere.

$$\boxed{(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2}$$

Ex: The describe the region defined by the inequalities

$$x^2 + y^2 + z^2 \leq 4 \qquad\qquad x^2 + y^2 \geq 1$$

This looks like a solid ball of radius 2, centered at the origin with a whole of radius 1 drilled through it along the z-axis.

12.2 Vectors

Definition 2. *A vector is an object with direction and magnitude. There is one exception to this definition, the zero vector, $\vec{0}$, which has magnitude 0 has no specified direction.*

Suppose a particle moves from a point A to a point B along a straight line. Then the displacement vector, written $\vec{AB}$, can be visualized as an arrow from A to B, visually:

If the points have coordinates $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ we can represent $\vec{AB}$ as

$$\vec{AB} = B - A = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

(this works for points in $\mathbb{R}^2$ as well)

12.2.1 Vector Operations

(Everything here is written for vectors in $\mathbb{R}^2$, but works in $\mathbb{R}^3$ as well)

Vector Addition $\vec{u}+\vec{v}$ - Place the tail of $\vec{v}$ on the tip of $\vec{u}$ then $\vec{u}+\vec{v}$ starts at the tail of $\vec{u}$ and ends at the tip of $\vec{v}$

If $\vec{u}=\langle u_1,u_2\rangle$ and $\vec{v}=\langle v_1,v_2\rangle$ then

$$\vec{u}+\vec{v}=\langle u_1+v_1,u_2+v_2\rangle$$

Negative $-\vec{v}$ - $-\vec{v}$ points in the opposite direction

$$-\vec{v}=\langle -v_1,-v_2\rangle$$

Scalar Multiplication $c\vec{v}$ - Scale the size of $\vec{v}$ by $|c|$. If $c<0$ then point $\vec{v}$ in the other direction

$c \in \mathbb{R}$ then

$$c\vec{v} = \langle cv_1, cv_2 \rangle$$

Vector Subtraction $\vec{u} - \vec{v}$ - Put the vectors tail to tail then $\vec{u} - \vec{v}$ is from the head of $\vec{v}$ to the head of $\vec{u}$.

$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$$

12.2.2 Magnitude of a Vector

In $\mathbb{R}^3$, $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Algebraic Properties of Vectors:

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(cd)\vec{a} = c(d\vec{a})$
7. $1\vec{a} = \vec{a}$

Given any vector $\vec{v} = \langle a, b, c \rangle$, using the rules above, we can write

$$\vec{v} = \langle a, b, c \rangle = a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

where $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \langle 0, 0, 1 \rangle$ are called standard basis vectors in $\mathbb{R}^3$ (likewise, $\hat{i} = \langle 1, 0 \rangle$ and $\hat{j} = \langle 0, 1 \rangle$ are the standard basis vectors for $\mathbb{R}^2$). The coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$ are called the components of $\vec{v}$.

Definition 3. *A unit vector is a vector of magnitude* 1. *(I will usually denote unit vectors with a hat instead of an arrow.)*

Given a vector $\vec{v} \neq \vec{0}$, one can find the unit vector in the direction of $\vec{v}$ by multiplying by $\frac{1}{||\vec{v}||}$, i.e.

$$\hat{v} = \frac{1}{||\vec{v}||}\vec{v}$$

is a unit vector in the direction of $\vec{v}$. Given a vector's magnitude and direction (angle it makes with positive x-axis) we can recover the vector: If $\vec{v}$ is the vector, $||\vec{v}||$ its magnitude and direction θ, $\vec{v}$ can be written:

$$\vec{v} = ||\vec{v}||\cos\theta\,\hat{i} + ||\vec{v}||\sin\theta\,\hat{j}$$

Of course, this is only true for 2 dimensional vectors. The procedure is a bit different in higher dimensions.

12.2.3 An Application

Ex: Suppose we have a 100kg suspended from the ceiling as depicted:

Using $g = 9.8\frac{m}{s^2}$ for acceleration due to gravity, find the tension in each cable.

Let $\vec{T}_1$ and $\vec{T}_2$ denote the tensions in the left and right cables, resp. Let $\vec{w}$ denote the weight vector. Then $\vec{w} = \langle 0, -980\rangle$. By Newton's 3rd law the sum of $\vec{T}_1$, $\vec{T}_2$, and $\vec{w}$ must be $\vec{0}$ since the weight is not in motion, i.e., $\vec{T}_1 + \vec{T}_2 + \vec{w} = \vec{0}$. In components we have 2 equations:

$$\begin{cases} ||\vec{T}_1||\cos 60^\circ + ||\vec{T}_2||\cos 30^\circ + 0 = 0 \\ ||\vec{T}_1||\sin 60^\circ + ||\vec{T}_2||\sin 30^\circ - 980 = 0 \end{cases}$$

then

$$\begin{cases} -\frac{1}{2}||\vec{T}_1|| + \frac{\sqrt{3}}{2}||\vec{T}_2|| = 0 \\ \frac{\sqrt{3}}{2}||\vec{T}_1|| + \frac{1}{2}||\vec{T}_2|| - 980 = 0 \end{cases}$$

so

$$\begin{cases} ||\vec{T}_1|| = \sqrt{3}||\vec{T}_2|| \\ \sqrt{3}||\vec{T}_1|| + ||\vec{T}_2|| = 1960 \end{cases}$$

Plugging the first into the second we have

$$3||\vec{T}_2|| + ||\vec{T}_2|| = 4||\vec{T}_2|| = 1960$$

So $||\vec{T}_2|| = 490$ then $||\vec{T}_1|| = 490\sqrt{3}$

12.3 Dot Product

We've discussed how to add, subtract, and multiply vectors by a scalar, but what about multiplying vectors? Should it produce a number, or a vector? This first product will produce a scalar:

Definition 4. *For* $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$, *the dot product of* $\vec{u}$ *and* $\vec{v}$ *is*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The dot product is sometimes called a scalar or inner product. (The dot product for 2D vectors is defined similarly.)

12.3.1 Properties of the Dot Product

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors and c a scalar.

1. $\vec{a} \cdot \vec{a} = ||\vec{a}||^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Suppose the angle between two vectors $\vec{u}$ and $\vec{v}$ is θ, then another interpretation of the dot product is:

$$\boxed{\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}|| \cos\theta}$$

This can be reversed to find the angle between two vectors $\vec{u}$ and $\vec{v}$

$$\boxed{\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}\right)}$$

Two vectors are called perpendicular or orthogonal if their dot product is 0 (i.e. $\theta = 90°$)

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$$

12.3.2 Projections

Let's say we have two vectors $\vec{u}$ and $\vec{v}$ as such

A question we could ask is "how much does $\vec{v}$ point in the direction of $\vec{u}$?" or "what is the piece of $\vec{v}$ in the $\vec{u}$-direction?"

The answer to the first question is called the scalar projection of $\vec{v}$ onto $\vec{u}$: $\text{comp}_{\vec{u}}\vec{v}$

Trigonometry tells us $\text{comp}_{\vec{u}}\vec{v} = ||\vec{v}||\cos\theta$. Recall that $\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta$, so

$$\boxed{\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||}}$$

(Notice that this number is negative if $\theta > 90°$)

The answer to the second question is the vector which is the "shadow" of $\vec{v}$ on $\vec{u}$:

It is called the vector projection of $\vec{v}$ onto $\vec{u}$.

This vector is parallel to $\vec{u}$ and its length is $\text{comp}_{\vec{u}}\vec{v}$ so a formula for it is

$$\text{proj}_{\vec{u}}\vec{u} = (\text{comp}_{\vec{u}}\vec{v})\frac{\vec{u}}{||\vec{u}||} = \left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||}\right)\frac{\vec{u}}{||\vec{u}||}$$

so

$$\boxed{\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\right)\vec{u}}$$

Ex: Find the vector projection of $\vec{v} = \langle 0, 1, \frac{1}{2}\rangle$ onto $\langle 2, -1, 4\rangle$.

$$\begin{aligned}
\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\right)\vec{u} &= \left(\frac{(2)(0)+(-1)(1)+(4)(1/2)}{(\sqrt{(2)^2+(-1)^2+(4)^2})^2}\right)\langle 2,-1,4\rangle \\
&= \left(\frac{0-1+2}{4+1+16}\right)\langle 2,-1,4\rangle \\
&= \frac{1}{21}\langle 2,-1,4\rangle
\end{aligned}$$

12.3.3 An Application: Work

Let's say a constant force $\vec{F}$ moves an object from the point P to the point Q. The displacement vector of the object is $\vec{d} = \vec{PQ}$. The amount of work $\vec{F}$ does in moving the object is the product of the component of $\vec{F}$ in the direction of $\vec{d}$ (i.e. $\text{comp}_{\vec{d}}\vec{F}$) and the displacement distance (i.e. $||\vec{d}||$). So, if θ is the angle between $\vec{F}$ and $\vec{d}$, we have

$$\text{Work} = \text{comp}_{\vec{d}}\vec{F}||\vec{d}|| = (||\vec{F}||\cos\theta)||\vec{d}|| = \vec{F}\cdot\vec{d}$$

Example A child pulls a red wagon a distance of $200m$ by exerting a force of $100N$ at $20°$ above the horizontal. How much work has the child done in moving the wagon?

$$\begin{aligned} W &= (||\vec{F}||\cos\theta)||\vec{d}|| = ((100\cos 20°)N)(200m) \\ &= 20000\cos 20°\,J \\ &\approx 18.794kJ \end{aligned}$$

12.4 Cross Product

Suppose we are given two vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = v_1, \vec{v}_2, v_3$. We would like to create a new vector $\vec{w} = \langle w_1, w_2, w_3\rangle$ out of them so that $\vec{u}, \vec{v} \perp \vec{w}$. The desired conditions give us two equations:

$$\begin{cases} \vec{u}\cdot\vec{w} = 0 \\ \vec{v}\cdot\vec{w} = 0 \end{cases}$$

This actually has a whole family of solutions, one of which is

$$\vec{w} = \langle u_2v_3 - v_3u_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$$

$\vec{w}$ is called the cross product of $\vec{u}$ and $\vec{v}$ and is written $\vec{u}\times\vec{v}$. We have a simpler way of computing the cross products than solving the above system or memorizing the above formula. It uses determinants. The cross product is also called the vector product.

12.4.1 Determinants

For a 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 matrix

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \end{aligned}$$

Using the unit vector notation we can write the cross product as

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

⚠Whereas the dot product can be taken any two vectors of the same dimension, the cross product only makes sense in dimension 3.

Ex: Find the cross product of $\vec{u} = \langle 1, 3, -2\rangle$ and $2, \vec{4}, 6$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 2 & 4 & 6 \end{vmatrix} = \hat{i}(18 - (-8)) - \hat{j}(6 - (-4)) + \hat{k}(4 - 6)$$
$$= \langle 26, -10, -2\rangle$$

Before, to check whether two nonzero vectors are parallel we need to find a constant c such that $\vec{u} = c\vec{v}$.

The cross product gives us an easier way.

Theorem 5. *Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$*

Proof. If $\vec{u}$ is parallel to $\vec{v}$, then $\vec{u} = c\vec{v}$ for some $c \in \mathbb{R}$. So $\vec{u} = c\vec{v} = \langle cv_1, cv_2, cv_3\rangle$, thus

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ cv_1 & cv_2 & cv_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= \hat{i}(cv_2v_3 - cv_3v_2) - \hat{j}(cv_1v_3 - cv_3v_1) + \hat{k}(cv_1v_2 - cv_2v_2)$$
$$= \vec{0}$$

□

Theorem 6. *If θ is the angle between $\vec{u}$ and $\vec{v}$ (so $0 \le \theta \le \pi$), then*

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{v}||\sin\theta$$

This theorem actually also has a nice geometrical application: Given two vectors $\vec{u}$ and $\vec{v}$, we get the parallelogram that they span

the area of which is $A = ||\vec{u}||\,||\vec{v}||\sin\theta = ||\vec{u} \times \vec{v}||$.

Ex. Find the area of the triangle with vertices $P = (0, 0, -3)$, $Q = (4, 2, 0)$, and $R = (3, 3, 1)$

Say the points are arranged as

Notice that the triangle $\triangle PQR$ has half the area of the parallelogram spanned by $\vec{PQ}$ and $\vec{PR}$. So,

$$\text{Area of } \triangle PQR = \frac{1}{2}||\vec{PQ} \times \vec{PR}||$$

$\vec{PQ} = Q - P = \langle 4, 2, 3\rangle$ and $\vec{PR} = \langle 3, 3, 4\rangle$ so

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\left|\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix}\right|\right| \\
&= \frac{1}{2}||(8-9)\hat{i} - (16-9)\hat{j} + (12-6)\hat{k}|| \\
&= \frac{1}{2}||\langle -1, -7, 6\rangle|| \\
&= \frac{1}{2}\sqrt{1+49+36} = \frac{1}{2}\sqrt{86}
\end{aligned}$$

Using the properties of the cross product we have so far, we have the following

$$\begin{array}{ccc}
\hat{i} \times \hat{j} = \hat{k} & \hat{j} \times \hat{k} = \hat{i} & \hat{k} \times \hat{i} = \hat{j} \\
\hat{j} \times \hat{i} = -\hat{k} & \hat{k} \times \hat{j} = -\hat{i} & \hat{i} \times \hat{k} = -\hat{j}
\end{array}$$

This can be remembered as a cyclic property

Moving in the direction of the arrows, no problem, moving against the arrows creates a minus sign in the answer.
Notice that this establishes that $\times$ is not commutative. Furthermore

$$(\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \hat{j} = \vec{0} \qquad \hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$$

meaning $\times$ is not even associative!

So, what properties are true?

12.4.2 Properties of the Cross Product

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors and c a scalar. Then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

12.4.3 Triple Product

Given 3 vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ the triple scalar product, is the product $\vec{u} \cdot (\vec{v} \times \vec{w})$, a scalar, and can be computed as a determinant with the 3 vectors as rows:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

A valid question to ask is "what is the purpose of this product?" The point is the following:
Just as 2 non-parallel, non-zero vectors span a parallelogram, 3 such vectors (in this they need to be pairwise non-parallel, which means non-coplaner) will span a parallelepiped:

The volume of a parallelepiped is Vol $= A \cdot h$ where A is the area of the base and h is the height. We already know $A = ||\vec{b} \times \vec{c}||$. We can find h with a little geometry:

so $h = ||\vec{a}|| \, |\cos\theta|$ (we need to use $|\cos\theta|$ in case $\theta > \frac{\pi}{2}$). This means that $\text{Vol} = A \cdot h = ||\vec{b} \times \vec{c}||(||\vec{a}|| \, |\cos\theta|)$. This is equivalent to

$$\text{Vol} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

The triple product has another use: checking whether 3 vectors are coplaner. Think about it geometrically. If the 3 vectors are coplanar, then the volume of the parallelepiped should be 0 since there is no "thrid direction". This gives us:

Three nonzero vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar if and only if $\vec{u}\dot{(}\vec{v} \times \vec{w}) = 0$.

Ex: Determine whether $\vec{u} = \langle 1, 5, -2 \rangle$, $\vec{v} = \langle 3, -1, 0 \rangle$, and $\vec{w} = \langle 5, 9, -4 \rangle$ are coplanar.

$$\begin{aligned} \vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} \\ &= 1(4-0) - 5(-12-0) - 2(27+5) \\ &= 4 + 60 - 64 \\ &= 0 \end{aligned}$$

So these vectors are coplanar.

12.4.4 An Application

Torque is created by applying a force to an object at a point given by a position vector, for example using a wrench to tighten a bolt. Torque is a measure of the tendency of the object to rotate about a pivot point (from which the position vector radiates). If the position vector is $\vec{r}$ and the force is $\vec{F}$, the torque vector is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$||\vec{\tau}|| = ||\vec{r}||\,||\vec{F}||\sin\theta$$

12.5 Lines and Planes

Let's look back at how we describe a line in the plane: we use the slope (read: direction) of the line and a point on the line:

$$y - y_0 = m(x - x_0)$$

The slope, $m = \frac{\text{rise}}{\text{run}}$, and notice we can encode that as a vector: $\vec{v} = \langle \text{run}, \text{rise} \rangle$ as follows:

We can see then that any multiple of $\vec{v}$ starting at (x_0, y_0) points to a point on l. This gives us the vector equation for the line:

$$\vec{l} = \vec{P}_0 + t\vec{v}\,, \quad \vec{P}_0 = \langle x_0, y_0 \rangle\,, \quad \vec{l} = \langle x, y \rangle$$

If we use $\vec{v} = \langle l, m \rangle$, then

$$\langle x, y \rangle = \vec{l} + \vec{P}_0 + t\vec{v} = \langle x_0, y_0 \rangle + \langle t, mt \rangle$$

so

$$\begin{cases} x = x_0 + t \\ y = y_0 + mt \end{cases} \overset{m \neq 0}{\Longrightarrow} \begin{cases} t = x - x_0 \\ t = \frac{1}{m}(y - y_0) \end{cases} \Longrightarrow \frac{1}{m}(y - y_0) = x - x_0$$
$$\Longrightarrow y - y_0 = m(x - x_0)$$

which should look familiar.

In 3 dimensions, the equation for a line looks exactly the same:

direction vector:	$\vec{v} = \langle a, b, c \rangle$
position vector of point on line:	$\vec{P}_0 = \langle x_0, y_0, z_0 \rangle$

The vector equation of a line is then

$$\vec{l} = \vec{P}_0 + t\vec{v}$$

If we write this out:

$$\langle x, y, z \rangle = \vec{l} = \vec{P}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

We can then separate this into 3 equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

called the parametric equations of the line.

Ex: Find the vector and parametric equations for the line passing through $(-2, 4, 0)$ and $(1, 1, 1)$

First, we need a direction vector for the line. If $P = (-2, 4, 0)$ and $Q = (1, 1, 1)$, a direction vector is $\vec{v} = \vec{PQ} = \langle 3, -3, 1 \rangle$. So a vector equation for the line is

$$\vec{l} = \vec{0P} + t\vec{v} = \langle -2, 4, 0 \rangle + t\langle 3, -3, 1 \rangle$$

From this we can read off the parametric equations

$$\begin{cases} x = -2 + 3t \\ y = 4 - 3t \\ z = 0 + t \end{cases}$$

Just as above, we can combine the parametric equations

$$\begin{array}{ccc} x = x_0 + at & y = y_0 + bt & z = z_0 + ct \\ \downarrow a \neq 0 & \downarrow b \neq 0 & \downarrow c \neq 0 \\ t = \frac{x - x_0}{a} & t = \frac{y - y_0}{b} & t = \frac{z - z_0}{c} \end{array}$$

Combining these together we get the Symmetric Equations of a line:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.}$$

It could happen that one (or even 2) of the components of $\vec{v}$ are zero. An example is if $a = 0$, then the symmetric equations would take the form

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Ex. Find the symmetric equations of the line in the previous example.

$$\frac{x-(-2)}{3} = \frac{y-4}{-3} = \frac{z-0}{1}$$

Simplifying this we have

$$\frac{x+2}{3} = -\left(\frac{y-4}{3}\right) = z$$

Sometimes we don't want a whole line, but just a line segment. If we already have an equation for the whole line, we can just restrict the parameter t to start at the first point and end at the second. So you end up with something like this:

$$\vec{l}(t) = \vec{P}_0 + t\vec{v}, \quad a \le t \le b.$$

The quickest way to parametrize a line segment, however, is as follows:

If we want the line segment from P to Q it's parametrized by:

$$\vec{l}(t) = (1-t)0\vec{P} + t0\vec{Q}, \quad 0 \le t \le 1$$

In the plane, we know two lines are either parallel or they intersect. Lines in space, however, can be both non-parallel and non-intersecting. These are called skew lines.

Ex: Show that the lines:

$$L_1: \ x = 3+2t, \ y = 4-t, \ z = 1+3t$$
$$L_2: \ x = 1+4s, \ y = 3-2s, \ z = 4+5s$$

are skew.

This is done in two steps. First, we show they're not parallel. This is as easy as checking if their direction vectors are parallel. The direction vectors are: $\vec{v}_1 = \langle 2, -1, 3\rangle$ for L_1 and $\vec{v}_2 = \langle 4, -2, 5\rangle$ for L_2. It's easy to see that one is not a multiple of the other, so the lines are not parallel. To see if the lines intersect, we set them equal to each other and try to solve the system:

$$\begin{cases} x = 3+2t = 1+4s \\ y = 4-t = 3-2s \\ z = 1+3t = 4+5s \end{cases}$$

implies

$$\begin{cases} 2t-4s = -2 \\ -t+2s = -1 \\ 3t-5s = 4 \end{cases}$$

Now the first equation is equivalent to $t-2s = -1$ and the second is equivalent to $t-2s = 1$ which contradict each other. Thus the system has no solution, so the lines do not intersect. Meaning, the lines are skew.

The natural generaization of a line is a plane. We again need two pieces of information to get the equation of a plane:

1. A point $P_0 = (x_0, y_0, z_0)$ in the plane
2. A vector normal (perpendicular) to the plane $\vec{n} = \langle a, b, c\rangle$

$P = (x, y, z)$ is any point in the plane

How does this give us a plane?

Notice how $\vec{n} \perp \vec{P_0P}$ for any point P in the plane. So, an equation for the plane is

Vector equation of the plane Π	$\vec{n} \cdot \vec{P_0P} = 0$

Filling in $\vec{n} = \langle a, b, c \rangle$ and $\vec{P_0P} = x - x_0, y - y_0, z - z_0$ gives the scalar equation of the plane:

$$\begin{aligned} \vec{n} \cdot \vec{P_0P} &= \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

Sometimes this is written as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.
Ex: Find an equation for the plane passing through $P = (0, 1, 1)$, $Q = (1, 0, 1)$, and $R = (1, 1, 0)$.

We already have a point in the plane (3 even!), so we just need the normal vector notice we can make two vectors in the plane starting from P: $\vec{PQ}$ and $\vec{PR}$

$$\vec{PQ} = \langle 1-0, 0-1, 1-1\rangle = \langle 1, -1, 0\rangle$$
$$\vec{PR} = \langle 1-0, 1-1, 0-1\rangle = \langle 1, 0, -1\rangle$$

Now we can use these two vectors in the plane (which are not parallel!) to make a normal vector by taking their corss product:

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle$$

So, an equation is:

$$\begin{aligned} \vec{n} \cdot \langle x-0, y-1, z-1\rangle &= \langle 1,1,1\rangle \cdot \langle x, y-1, z-1\rangle \\ &= x + (y-1) + (z-1) = 0 \end{aligned}$$

or equivalently

$$x + y + z = 2$$

Now, we have two kinds of objects in space: lines and planes. We already know the situation for two lines (intersecting, parallel, or skew), so how about the other pairs? Let's start wit ha line and a plane. Two things can happen: they're parallel or they intersect.

Ex: Does the line

$$L: \; x = 3+3t, \;\; y = t, \;\; z = -2+4t$$

intersect the plane $x+y+z=2$? If so, where?

If the line intersects the plane, we can plug the line into the equation for the plane and solve for a t value.

$$x + y + z = (3+3t) + (t) + (-2+4t) = 1 + 8t = 2$$

Solving this gives $t = \frac{1}{8}$. So, they do intersect and the point of intersection is

$$(x, y, z) = (3 + 3\left(\frac{1}{8}\right), \frac{1}{8}, -2 + 4\left(\frac{1}{8}\right)) = \left(\frac{27}{8}, \frac{1}{8}, \frac{-3}{2}\right)$$

How, now, about 2 planes? It's possible they're parallel (to check this, check if their normal vectors are parallel). More likely, though, they'll intersect. As you can probably see, they don't intersect in a point, but a line!

Ex: Do the planes $2x - 3y + 4z = 5$ and $x + 6y + 4z = 3$ intersect? If so, what is the angle of their intersection? Also, give an equation for their line of intersection.

The normal vectors of the planes are

$$\vec{n}_1 = \langle 2, -3, 4\rangle \qquad \vec{n}_2 = \langle 1, 6, 4\rangle$$

which can easily be seen to not be parallel since one is not a multiple of the other. So the planes are not parallel, thus they intersect. The angle of intersection is the same as the angle between their normal vectors:

$$\theta = \arccos\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1||\,||\vec{n}_1||}\right) = \arccos\left(\frac{(2)(1)+(-3)(6)+(4)(4)}{(\sqrt{4+9+16})(\sqrt{1+36+16})}\right) = \arccos(0) = \frac{\pi}{2}$$

(This actually means the planes are perpendicular!)
Now, for the line of intersection, we need a point and a direction vector. Let's start with the direction. The line lies in both planes, so its direction vector $\vec{v}$ must be perpendicular to both $\vec{n}_1$ and $\vec{n}_2$ since its parallel to both planes. We have a trick for creating a vector orthogonal to two given vectors: the cross product.

$$\begin{aligned}\vec{v} = \vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 6 & 4 \end{vmatrix} \\ &= \langle -12-24, -(8-4), 12-(-3)\rangle \\ &= \langle -36, -4, 15\rangle\end{aligned}$$

We may as well choose $\vec{v} = \vec{n}_1 \times \vec{n}_2$. Now, for a point on the line, we just need to find a point on both planes, that is, a solution to both $2x-3y+4z=5$ and $x+6y+4z=3$. We have two equations and three variables, so we'll have to choose a avalue for one of them, say $z=0$. Then, we need to solve the system:

$$\begin{cases} 2x-3y=5 \\ x+6y=3 \end{cases}$$

Two times the first plus the second yields: $5x=13$, so $x=\frac{13}{5}$. So plugging this back into the second we have $6y = 3 - \frac{13}{5} = \frac{2}{5}$, so $y = \frac{1}{15}$. This means the point $(\frac{13}{5}, \frac{1}{15}, 0)$ is on the line.
The symmetric equations for this line then are

$$\frac{x-\frac{13}{5}}{-36} = \frac{y-\frac{1}{15}}{-4} = \frac{z}{15}$$

Consider the following situation:

We're given a plane Π and a point P. How can we find the distance, D, from the plane to the point?

First, we know that the shortest path from the plane to the point is a straight line perpendicular to the plane, that is a line in the direction of $\vec{n}$, the normal vector to Π. Notice that if we take some point P_0 on Π and connect it to P, we get a vector connecting Π to P, and, moreover, if we project $\vec{P_0P_1}$ onto $\vec{n}$, we get a vector perpendicular to Π which starts on Π and ends at P. The length of this vector, then, is precisely D, i.e.

$$\boxed{D = ||\text{proj}_{\vec{n}} \vec{P_0P_1}|| = |\text{comp}_{\vec{n}} \vec{P_0P_1}|}$$

If $\vec{n} = \langle a, b, c \rangle$, $P_0 = (x_0, y_0, z_0)$, and $P_1 = (x_1, y_1, z_1)$, then

$$D = |\text{comp}_{\vec{n}} \vec{P_0P_1}| = \frac{|\vec{n} \cdot \vec{P_0P_1}|}{||\vec{n}||} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

If the plane is written as $ax + by + cz + d = 0$ then

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Let's see how this can be used to answer a related question.
Ex. Find the distance between the parallel planes $x - 4y + 2z = 0$ and $2x - 8y + 4z = -1$

Our situation looks as follows: If we forgot everything except P_1 from the top plane, we've reduced the problem to the distance between a point and a plane. First, we need to find a P_1 (it doesn't matter which plane P_1 is on, as long as P_0 is on the other one). Let's take P_1 on the second plane. Any point works, so the easiest way to get one is to make two components equal to zero, e.g., take $P_1 = (-\frac{1}{2}, 0, 0)$. A point on the other plane is $P_0 = (0, 0, 0)$. A normal vector to the planes is $\vec{n} = \langle 1, -4, 2 \rangle$, so

$$D = |\text{comp}_{\vec{n}} \vec{P_0P_1}| = \frac{|\vec{n} \cdot \vec{P_0P_1}|}{||\vec{n}||} = \frac{|(1)\left(-\frac{1}{2}\right) + (-4)(0) + (2)(0)|}{\sqrt{1+16+4}} = \frac{\frac{1}{2}}{\sqrt{21}} = \frac{1}{2\sqrt{21}}$$