13 Chapter 13

13.1 Vector Valued Functions and Space Curves

A vector-valued function is a function whose output is a vector. We have already encountered one: the vector equation for a line

$$\vec{l}(t) = \vec{P}_0 + t\vec{v}$$

More generally, they will have the form

$$\begin{aligned}\vec{r}(t) &= \left\langle f(t), g(t), h(t) \right\rangle \\ &= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}\end{aligned}$$

The input variable (in this case it's t) is called the parameter. Since $\vec{r}$ is a function, we can ask about its domain. The domain of a vector-valued function is the "intersection" of the domains of its components functions, that is, the values common to the domains of each of f, g, and h.
Ex: What is the domain of $\vec{r}(t) = \left\langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \right\rangle$?

First, we find the domains of each of the component functions:

function	$f(t) = \sqrt{4-t^2}$	$g(t) = e^{-3t}$	$h(t) = \ln(t+1)$
domain	$-2 \le t \le 2$	$-\infty < t < \infty$	$-1 < t < \infty$

The t-values in common to each of these are $-1 < t \le 2$. So the domain is $(-1, 2]$.

As with normal functions, we can take limits of vector-valued functions:
If $\vec{r}(t) = \left\langle f(t), g(t), h(t) \right\rangle$, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

And this leads us to the definition of continuity for vector valued functions:

Definition 1. *A vector valued function $\vec{r}(t)$ is called continuous at a if*

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

Let's look at some exampls of vector-valued functions.
Ex:

i) $\vec{r}(t) = \langle \sin t, t \rangle$ ii) $\vec{r}(t) = \langle 3\cos t, \sin t \rangle$
iii) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ iv) $\langle t, \sin t, 2\cos t \rangle$
v) $\vec{r}(t) = \langle t\cos t, t\sin t, t \rangle$

i)

ii)

Notice that $x = 3\cos t$ and $y = \sin t$ satisfies $\frac{x^2}{2} + y^2 = 1$ the equation of an ellipse!

iii) With 3D space curves, it's often useful to find a surface that your curve sits on. In this case we have

$$\vec{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle = \langle \cos t, \sin t, t \rangle$$

so $[x(t)]^2 + [y(t)]^2 = 1$. This means the curve sits on the cylinder and climbs up it as t increases:

iv) Here, $[y(t)]^2 + \frac{[z(t)]^2}{4} = 1$, so the curve sits on the elliptic cylinder $y^2 + \frac{z^2}{4} = 1$ (it opens along the x-axis).
v) This one satisfies

$$[x(t)]^2 + [y(t)]^2 = t^2(\cos^2 t + \sin^2 t) = t^2 = [z(t)]^2$$

meaning it sits on the cone $x^2 + y^2 = z^2$

Often of importance is the intersection of surfaces. The result is generically a curve (i.e., the intersection of two planes is a line, as we know). It is important to be able to quantify these "curves of intersection" by parameterizing them (finding a vector-valued function whose image is the curve).
Ex. Find a vector function representing the intersection of $z = x^2$ and $x^2 + y^2 = 4$.

We know that whatever the vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is, it must satisfy the equations of the surfaces since it lies on both of them, that is, we must have

$$z(t) = [x(t)]^2 \qquad \text{and} \qquad [x(t)]^2 + [y(t)]^2 = 4\ .$$

The second equation suggests taking

$$x(t) = 2\cos t \qquad \text{and} \qquad y(t) = 2\sin t$$

and first then gives $z(t) = 4\cos^2 t$.
Thus the vector equation is

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\cos^2 t \rangle\ .$$

13.2 Derivatives and Integrals

What were the main applications of limits back in Calc I? Differentiation and integration!

Definition 2. *The derivative of a vector valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is*

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)h}{h} = \langle f'(t), f'(t), h'(t) \rangle$$

Recall that the derivative of a function gave us the slope of its tangent line. For vector-valued functions, the derivative gives us a tangent vector (pointing in the direction of increasing t-values). This vector can be used as a direction vector for the tangent line.

$\vec{r}\,'(a)$ is the tangent vector to $\vec{r}(t)$ at $t = a$ (provided $\vec{r}\,'(a) \neq \vec{0}$). A very important variation of the tangent vector is the unit tangent vector:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$$

This will be important in the next section.
Ex: Let $\vec{r}(t) = \langle t\cos t, t, t\sin t\rangle$. Find: i) $\vec{r}\,'(t)$ ii) $\vec{T}(t)$ iii) an equation for the tangent line to $\vec{r}(t)$ at $t = \pi$.

i) $\vec{r}\,'(t) = \langle \cos t - t\sin t, 1, \sin t + t\cos t\rangle$
ii) First we need $||\vec{r}\,'(t)||$:

$$\begin{aligned} ||\vec{r}\,'(t)|| &= \sqrt{(\cos t - t\sin t)^2 + 1 + (\sin t + t\sin t)^2} \\ &= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 + 1 + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} \\ &= \sqrt{1 + t^2 + 1} = \sqrt{t^2 + 2} \end{aligned}$$

So,

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = \frac{1}{\sqrt{t^2+2}}\langle \cos t - t\sin t, 1, \sin t + t\cos t\rangle$$

iii) A direction vector for the tangent line at $t = \pi$ is

$$\vec{r}\,'(\pi) = \langle \cos\pi - \pi\sin\pi, 1, \sin\pi + \pi\cos\pi\rangle = \langle -1, 1, -\pi\rangle$$

(note that we could have also used $\vec{T}(\pi)$)
A point on the tangent line is $\vec{r}(\pi) = \langle -\pi, \pi, 0\rangle$, so an equation for the tangent line is

$$\vec{l}(s) = \vec{r}(\pi) + s\vec{r}\,'(\pi) = \langle -\pi - s, \pi + s, \pi s\rangle$$

13.2.1 Properties of Derivatives

Let $\vec{u}(t)$, $\vec{v}(t)$ be vector functions, c a constant, and $f(t)$ a scalar function. Then

1. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$

2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}\,'(t)$

3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

4. $\frac{d}{dt}[\vec{u}(t)\cdot\vec{u}(t)] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

5. $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

6. $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

The first two of these establish the linearity of the derivative of vector functions. The next three are product rules of vector functions. The final is the chain rule of vector functions. There is a useful consequence of #4. Suppose $\vec{r}'(t) \neq \vec{0}$ then

$$\begin{aligned}\frac{d}{dt}||\vec{r}(t)|| = \frac{d}{dt}\sqrt{\vec{r}(t)\cdot\vec{r}(t)} &= \frac{1}{2}\frac{\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)]}{\sqrt{\vec{r}(t)\cdot\vec{r}(t)}}\\ &= \frac{1}{2}\frac{2\vec{r}(t)\cdot\vec{r}'(t)}{||\vec{r}(t)||} = \frac{\vec{r}(t)\cdot\vec{r}(t)}{||\vec{r}(t)||}\end{aligned}$$

Finally, integration:

Definition 3. *The* <u>*definite integral*</u> *of* $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ *is*

$$\begin{aligned}\int_a^b \vec{r}(t)\,d &= \lim_{n\to\infty}\sum_{i=1}^{n}\vec{r}(t_i^*)\Delta t_i\\ &= \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\right\rangle\end{aligned}$$

The t_i^* *are chosen from the* i^th *piece of a partition of* $[a,b]$ *into n pieces. One can also define* <u>*indefinite integrals*</u>*.*

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt\right)\hat{i} + \left(\int g(t)\,dt\right)\hat{j} + \left(\int h(t)\,dt\right)\hat{k}$$

Ex: Find $\int \vec{r}(t)\,dt$ and $\int_0^2 \vec{r}(t)\,dt$ where $\vec{r}(t) = t\hat{i} - t^3\hat{k}$.

$$\begin{aligned}\int \vec{r}(t)\,dt &= \left(\int t\,dt\right)\hat{i} + \left(\int 0\,dt\right)\hat{j} + \left(\int -t^3\,dt\right)\hat{k}\\ &= \left(\frac{1}{2}t^2 + C_1\right)\hat{i} + C_2\hat{j} + \left(-\frac{1}{4}t^4 + C_3\right)\hat{k}\end{aligned}$$

$$\int_0^2 \vec{r}(t)\,dt = \left(\frac{1}{2}t^2\Big|_0^2\right)\hat{i} + 0\Big|_0^2\hat{j} + \left(-\frac{1}{4}t^4\Big|_0^2\right)\hat{k} = 2\hat{i} - 4\hat{k}$$

13.3 Arc Length and Curvature

In Calc II we found the arc length of a plane curve $x(t) = f(t)$, $y(t) = g(t)$, $a \le t \le b$ as

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_a^b \sqrt{dx^2 + dy^2} \end{aligned}$$

This was done by approximating the curve by straight lines. We can do the same thing for curves in $\mathbb{R}^3$. This leads to:

Definition 4. *The* <u>*arc length*</u> *of the curve* $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$ *i*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

There are some technical assumptions to this formula:

- $\vec{r}(t)$ does not cross itself between $t = a$ and $t = b$
- f', g', and h' must be continuous (i.e. $\vec{r}$ is C^1)

Notice that because

$$||\vec{r}\,'(t)|| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

we have that the arc length can be computed as

$$L = \int_a^b ||\vec{r}\,'(t)||\,dt$$

Ex: Find the arc length of

$$\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$$

where $-5 \le t \le 5$.

$$\vec{r}\,'(t) = \langle 1, -3\sin t, \sin t \rangle \; ||\vec{r}\,'(t)|| \qquad = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$$

So $L = \int_{-5}^{5} \sqrt{10}\,dt = 10\sqrt{10}$

A curve C need not have a unique representation by a vector function, in fact, none do. For example $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $1 \le t \le 2$ is also represented by $\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle$, $0 \le u \le \ln(2)$. $\vec{r}_1$ and $\vec{r}_2$ are called parametrizations of C. There is one particular parametrization we care about, and it is found as follows:

Suppose C is given by $\vec{r}(t)$, $a \le t \le b$, with $\vec{r}\,'$ continuous and $\vec{r}(t)$ traverses C exactly once. We can define the <u>arc length function</u>

$$s(t) = \int_a^t ||\vec{r}\,'(u)||\,du$$

which tells us the distance traveled at time t. Now, suppose we can solve this equation for t in terms of s. In other words invert $s(t)$ so that $t = t(s)$. Then:

Definition 5. *The* <u>*Arc Length Reparametrization*</u> *of* $\vec{r}(t)$ *is*

$$\vec{r} = \vec{r}(t(s)), \quad 0 \le s \le L$$

where L = *arc length of* $\vec{r}$ *from* $t = a$ *to* $t = b$.

Ex: Reparametrize $\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$, $-5 \le t \le 5$ with respect to arc length.

Firs, find the arc length function:

$$\begin{aligned} s = s(t) &= \int_{-5}^{t} ||\vec{t}'(u)||\, d = \int_{-5}^{t} \sqrt{10}\, du \\ &= \sqrt{10}t + 5\sqrt{10} \end{aligned}$$

Solving for t gives: $t = \frac{s-5\sqrt{10}}{\sqrt{10}} = t(s)$.
From the last example, $L = 10\sqrt{10}$, so the bounds on s are $0 \le s \le 10\sqrt{10}$, so the bounds on s, thus the reparametrization is:

$$\vec{r}(t(s)) = \left\langle \frac{s-5\sqrt{10}}{\sqrt{10}}, 3\cos\left(\frac{s-5\sqrt{10}}{\sqrt{10}}\right), 3\sin\left(\frac{s-5\sqrt{10}}{\sqrt{10}}\right)\right\rangle, \quad 0 \le s \le 10\sqrt{10}\,.$$

An interesting fact about arc length reparametrizations:

$$\frac{d\vec{r}}{ds} = \frac{d}{ds}(\vec{r}(t(s))) = \left(\frac{dt}{ds}\right)\vec{r}\,'(t(s)) = \frac{1}{||\vec{r}\,'(t)||}\vec{r}\,'(t(s))$$

So, $\left|\left|\frac{d\vec{r}}{ds}\right|\right| = 1$, that is, arc length reparametrizations always move with unit speed!

13.3.1 Curvature

Intuitively, curvature is a measure of how sharply a curve bends. Pictorally

has larger curvature at p than

does.

Definition 6. *A parametrization* $\vec{r}(t)$ *is called* <u>*smooth*</u> *on an interval* I *if* $\vec{r}\,'$ *is continuous on* I *and* $\vec{r}\,'(t) \neq \vec{0}$ *for any* $t \in I$. *A curve* C *is called* <u>*smooth*</u> *if it has a smooth parametrization.*

We quantify <u>curvature</u> as the rate of change of the unit tangent vector with respect to arc length. In symbols, the curvature of $\vec{r}$ is

$$\boxed{\kappa = \left|\left|\frac{d\vec{T}}{ds}\right|\right|}$$

Now $\frac{d\vec{T}}{ds}$ can often be messy to compute, however, we have a trick: by the chain rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \frac{d\vec{T}}{ds}||\vec{r}\,'(t)||$$

So, a more convenient formula for curvature is

$$\boxed{\kappa = \frac{||\vec{T}\,'(t)||}{||\vec{r}\,'(t)||}}$$

Ex: Find the curvature of a circle of radius a.

Parametrize it: $\vec{r}(t) = \langle -a\sin t, a\cos t\rangle$ then $\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle$, $||\vec{r}\,'(t)|| = a$. So,

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = \langle -\sin t, \cos t\rangle$$

Then $\vec{T}\,'(t) = \langle -\cos t, -\sin t\rangle$ and $||\vec{T}\,'(t)|| = 1$, so

$$\kappa(t) = \frac{1}{a}\,.$$

Even that formula is more effort than needed. Another is

$$\boxed{\kappa(t) = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}}$$

Find the curvature of $\vec{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t}\right\rangle$ at $(0,1,1)$.

First, find the relavent t value. Since $\vec{r}(0) = \left\langle 0, e^0, e^{-0} \right\rangle = \langle 0,1,1 \rangle$, the t value is $t = 0$. We need $\vec{r}'(t)$ and $\vec{r}''(t)$:

$$\vec{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle \qquad \vec{r}''(t) = \left\langle 0, e^t, e^{-t} \right\rangle$$

Now, we could either plug all of this into the formula for $\kappa(t)$ first, then plug in $t = 0$, or plug in $t = 0$ now, then compute $\kappa(0)$. We'll do the latter.

$$\vec{r}'(0) = \left\langle \sqrt{2}, 1, -1 \right\rangle \qquad \vec{r}''(0) = \langle 0,1,1 \rangle$$

So

$$\vec{r}'(0) \times \vec{r}''(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2} & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \left\langle 2, -\sqrt{2}, \sqrt{2} \right\rangle .$$

Then $||\vec{r}'(0)|| = \sqrt{2+1+1} = \sqrt{4}$ and $||\vec{r}'(0) \times \vec{r}''(0)|| = \sqrt{4+2+2} = \sqrt{8} = 2\sqrt{2}$. So

$$\kappa(0) = \frac{2\sqrt{2}}{2^3} = \frac{\sqrt{2}}{4}$$

In the special case of a plane curve $y = f(x)$, by parametrizing it as $\vec{r}(x) = \left\langle x, f(x) \right\rangle$ we get

$$\kappa(x) = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}} .$$

13.3.2 Frenet-Serret Frame "$T - N - B$ Frame"

This consists of 3 vectors derived from a parametrization, $\vec{r}(t)$: $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$. We already know one of them, the other two are

Unit Normal Vector (Requiring $||\vec{T}'(t)|| \neq 0$, equivalently $\kappa(t) \neq 0$)

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

Binormal Vector

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Since $||\vec{T}(t)|| = 1$, we have $\vec{T}(t) \cdot \vec{T}'(t) = 0$, so $\vec{T} \perp \vec{N}$. By definition of $\times$, $\vec{B} \perp \vec{T}$, $\vec{N}$, so the three vectors are all orthogonal to each other. Thus since $||\vec{T}|| = ||\vec{N}|| = 1$, we have $||\vec{B}|| = ||\vec{T} \times \vec{N}|| = ||\vec{T}|| ||\vec{N}|| \sin \frac{\pi}{2} = 1$, thus all of $\vec{T}$, $\vec{N}$, and $\vec{B}$ and hard to compute, so here's an alternate way:

$$\vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{||\vec{r}'(t) \times \vec{r}''(t)||} \qquad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t) .$$

$\vec{N}$ always points in the direction the curve is bending and $\vec{B}$ points orthoganal to the motion of the curve.

We can create some planes using $\vec{T}$, $\vec{N}$, and $\vec{B}$.

Normal Plane: This plane is perpendicular to $\vec{r}(t)$. It is determined by $\vec{N}$ and $\vec{B}$, and so has $\vec{T}$ as a vector

orthogonal to it.
Osculating Plane: This plane best captures the motion of the curve. It is determined by $\vec{T}$ and $\vec{N}$, and so has $\vec{B}$ as a vector perpendicular to it.
Rectifying Plane: This plane determined by $\vec{T}$ and $\vec{B}$. We won't bother with this one.

Ex: Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ for $\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$ and find equations for the normal and osculating planes at $(\pi/2, 0, 3)$.

Let's begin with computing $\vec{T}(t)$: $\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t\rangle$, $||\vec{r}\,'(t)|| = \sqrt{1+9\sin^2 t + 9\cos^2 t} = \sqrt{10}$. So

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle\ .$$

This is pretty tame, so we can differentiate it:

$$\vec{T}'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle$$

So

$$||\vec{T}'(t)|| = \sqrt{0^2 + \frac{9\cos^2 t}{10} + \frac{9\sin^2 t}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

Thus,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = \frac{\frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle}{\frac{3}{\sqrt{10}}} = \langle 0, -\cos t, -\sin t\rangle$$

Finally

$$\begin{aligned}\vec{B}(t) = \vec{T}(t)\times\vec{N}(t) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}}\sin t & \frac{3}{\sqrt{10}}\cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} \\ &= \left\langle \frac{3}{\sqrt{10}}\sin^2 t + \frac{3}{\sqrt{10}}\cos^2 t, \frac{1}{\sqrt{10}}\sin t, -\frac{1}{\sqrt{10}}\cos t\right\rangle \\ &= \frac{1}{\sqrt{10}}\langle 3, \sin t, -\cos t\rangle\end{aligned}$$

The t-value corresponding to $(\pi/2, 0, 3)$ is $t = \pi/2$. So, we have:

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}, 0\right\rangle \quad \text{and} \quad \vec{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\right\rangle$$

<u>Normal Plane</u>: Use $\vec{T}(\pi/2)$:

$$\left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}, 0\right\rangle \cdot \left\langle x - \frac{\pi}{2}, y-0, z-3\right\rangle = 0$$

<u>Osculating Plane</u>: Use $\vec{B}(\pi/2)$:

$$\left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\right\rangle \cdot \left\langle x - \frac{\pi}{2}, y-0, z-3\right\rangle = 0$$

A comment on the normal and osculating planes:
Recall that we only need a vector which is perpendicular to the plane to find an equation for it, in particular, the length of the vector doesn't matter. So, easier vectors to use are:
<u>Normal Plane</u>: use $\vec{r}\,'(t)$
<u>Osculating Plane</u>: use $\vec{r}\,'(t)\times\vec{r}\,''(t)$

13.4 Motion in Space

Suppose a particle moves along a trajectory $\vec{r}(t)$.
It's velocity is $\vec{v}(t) = \vec{r}'(t)$
acceleration is $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$
and speed is $||\vec{v}(t)||$, which I wil denote by v.
Ex: A particle has acceleration function

$$\vec{a}(t) = 4t\hat{\imath} + 6\sin t\hat{\jmath} + e^t\hat{k} .$$

If its initial velocity is $\vec{v}(0) = 3\hat{\jmath}$ and its initial position is $\vec{r}(0) = \vec{0}$, find its position function.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = (2t^2 + C_1)\hat{\imath} + (-6\cos t + C_2)\hat{\jmath} + (e^t + C_3)\hat{k}$$

$\vec{v}(0) = C_1\hat{\imath} + (-6 + C_2)\hat{\jmath} + (1 + C_3)\hat{k} = 3\hat{\jmath}$ meaning $C_1 = 0$, $C_2 = 9$, $C_3 = -1$ so

$$\vec{v}(t) = \int \vec{a}(t)\,dt = 2t^2\hat{\imath} + (9 - 6\cos t)\hat{\jmath} + (e^t - 1)\hat{k}$$

Now

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\frac{2}{3}t^3 + D_1\right)\hat{\imath} + (9t - 6\sin t + D_2)\hat{\jmath} + (e^t - t + D_3)\hat{k}$$

$\vec{r}(0) = D_1\hat{\imath} + D_2\hat{\jmath} + (1 + D_3)\hat{k} = \hat{0}$ meaning $D_1 = D_2 = 0$ and $D_3 = -1$ so

$$\vec{r}(t) = \frac{2}{3}t^3\hat{\imath} + (9t - 6\sin t)\hat{\jmath} + (e^t - t - 1)\hat{k} .$$

If the particle has mass m and acceleration $\vec{a}(t)$, the force it experiences is given by Newton's second law:

$$\vec{F}(t) = m\vec{a}(t) .$$

Ex: A projectile is fired with a muzzle speed 200 m/s and angle of elevation 60°. If the projectile is fired from a distance of $10m$ above ground level, what is the distance covered by the projectile? (All forces, except gravity, are assumed negligible.)

The only force acting on the projectile is gravity, so $\vec{F}(t) = m\vec{a}(t) = -mg\hat{j}$. This means $\hat{a}(t) = -g\hat{j}$. So

$$\vec{v}(t) = \int \vec{a}(t) = -mg\hat{j}\ .$$

This means $\vec{a}(t) = -g\hat{j}$. So, $\vec{v}(t) = \int \vec{a}(t)\,dt = C_1\hat{i} + (-gt + C_2)\hat{j}$. To get the initial velocity, we use the given information $v = 200 = ||\vec{v}(0)||$. So

$$\vec{v}_0 = (200\cos 60^\circ)\hat{i} + (200\sin 60^\circ)\hat{j} = 100\hat{i} + 100\sqrt{3}\hat{j}$$

meaning $C_1 = 100$ and $C_2 = 100\sqrt{3}$. Thus the velocity function is

$$\vec{v}(t) = 100\hat{i} + (100\sqrt{3} - gt)\hat{j}$$

The position function is

$$\vec{r}(t) = \int \vec{v}(t)\,dt = (100t + D_1)\hat{i} + (100\sqrt{3}t - \frac{1}{2}gt^2 + D_2)\hat{j}\ .$$

The initial position is $\vec{r}(0) = 10\hat{j}$, so $D_1 = 0$, $D_2 = 10$. Thus

$$\vec{r}(t) = 100t\hat{i} + (100\sqrt{3}t - \frac{1}{2}gt^2 + 10)\hat{j}\ .$$

The particle hits the ground when the $\hat{j}$-component is 0: $100\sqrt{3}t - \frac{1}{2}gt^2 + 10 = 0$ meaning

$$t = \frac{-100\sqrt{3} \pm \sqrt{30000 + 20g}}{-g}$$

We take the positive value of t

$$t = \frac{-100\sqrt{3} - \sqrt{30000 + 20g}}{-g} \approx 35.4$$

Plugging this in the $\hat{i}$-component gives the distance traveld: dist $\approx 100(35.4)m \approx 3540m$.

Recall that the motion of a curve is best captured by the osculating plane at any point. (After all, $\vec{B}(t) \perp \vec{r}'(t), \vec{r}''(t)$.) We aim to write the acceleration in terms of $\vec{T}(t)$ and $\vec{N}(t)$. Let's start with $\vec{T}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{\vec{v}(t)}{v(t)}$$

So $\vec{v}(t) = v(t)\vec{T}(t)$.
Take a derivative:

$$\vec{v}''(t) = v'(t)]\vec{T}(t) + v(t)\vec{T}'(t) = \vec{a}(t)$$

Now $\kappa(t) = \frac{||\vec{T}'(t)||}{||\vec{r}'(t)||} = \frac{||\vec{T}'(t)||}{v(t)}$ so $||\vec{T}'(t)|| = v(t)\kappa(t)$.
This allows us to write:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = \frac{\vec{T}'(t)}{v(t)\kappa(t)}$$

So

$$\vec{T}'(t) = v(t)\kappa(t)\vec{N}(t)\ .$$

Finally: $\vec{a}(t) = v'(t)\vec{T}(t) + (v(t))^2\kappa(t)\vec{N}(t)$.
This motivates the definitions: tangential component of acceleration $a_T = v'$ and normal component of acceleration $a_N = v^2\kappa$.
With a little work we see

$$a_T = \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{||\vec{r}'(t)||} \qquad a_N = \frac{||\vec{r}'(t)\times\vec{r}''(t)||}{||\vec{r}'(t)||}$$

A convenient fact:
$\vec{a} = a_T\vec{T} + a_N\vec{N}$ Since $\vec{T}\cdot\vec{N} = 0$ and $\vec{T}\cdot\vec{T} = \vec{N}\cdot\vec{N} = 1$.

$$||\vec{a}||^2 = \vec{a}\cdot\vec{a} = (a_T\vec{T} + a_N\vec{N})\cdot(a_T\vec{T} + a_N\vec{N}) = a_T^2 + a_N^2$$

So, $||\vec{a}|| = \sqrt{a_T^2 + a_N^2}$
Ex: Find the normal and tengential components of acceleration for a particle moving along the trajectory $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$

$$\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$
$$\vec{r}''(t) = \langle -\cos t, -\sin t, 0\rangle$$

$$\vec{r}'(t)\cdot\vec{r}''(t) = \sin t\cos t - \cos t\sin t + 0 = 0$$

So $a_T = 0$ meaning $||\vec{a}|| = \sqrt{a_T^2 + a_N^2} = \sqrt{a_N^2} = a_N$

$$a_N = ||\vec{a}(t)|| = ||\vec{r}''(t)|| = \sqrt{\cos^2 t + \sin^2 t + 0^2} = 1$$