15 Chapter 15

15.1 Double Integrals Over Rectangles

Let's recall how integrals are defined

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

15.1.1 Double Integrals

Let's start with a simple region: a rectangle.
Let $R = [a,b] \times [c,d]$ and let $f = f(x,y)$ contain R in its domain. We'll also assume for now that $f \geq 0$ on R. We start by cutting up the rectangle

In each subrectangle R_{ij} we choose a simple point (x_i^*, y_j^*), and over each R_{ij} construct a column of height $f(x_i^*, y_i^*)$. Adding up these volumes gives an approximation of the volume under f:

$$\text{Vol} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \left(f(x_i^*, y_j^*)\Delta x \Delta y \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

This is the <u>double Riemann sum</u>.
Now, of course, to get the actual volume we have to take finer and finer partitions ($\Delta x, \Delta y \to 0 \iff m, n \to \infty$)
So

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

Now, none of this required $f \geq 0$ on R, so we get the final definition:

Definition 1. *The* <u>*double integral*</u> *of f over the rectangle R is*

$$\int\int_R f(x,y)\,dA = \lim_{m,n\to\infty} f(x_i^*, y_j^*)\Delta A$$

if the limit exists.

15.1.2 Facts

1. If $f(x,y) \geq 0$, then the volume V of the solid which lies above R and below the surface $z = f(x,y)$ is
$$V = \int\int_R f(x,y)\,dA$$
2. $$\int\int_R [f(x,y) + g(x,y)]\,dA = \int\int_R f(x,y)\,dA + \int\int_R g(x,y)\,dA$$
3. $$\int\int_R cf(x,y)\,dA = c\int\int_R f(x,y)\,dA$$
4. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
$$\int\int_R f(x,y)\,dA \geq \int\int_R g(x,y)\,dA$$

15.2 Iterated Integrals

This will give us a way of actually computing double integrals. Using the definition, let's take the "m-limit" first. This amounts to integrating x first:

$$\begin{aligned}
\int\int_R f(x,y)\,dA &= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{i=1}^{m} f(x_i^*, y_j^*)\,\Delta x\right)\Delta y \\
&= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\int_a^b f(x, y_j^*)\,d dx\right)\Delta y \\
&= \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \\
&= \int_c^d \int_a^b f(x,y)\,dx\,dy
\end{aligned}$$

Doing the "n-limit" first would give integrating y first

$$\int\int_r f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx\,.$$

These are called <u>iterated integrals</u>. Notice that we are working from the inside out in these integrals. So, now we have the question of how to compute $\int_a^b f(x,y)\,dx$ or $\int_c^d f(x,y)\,dy$? The answer is <u>partial integration</u> which are performed analogusly to partial derivatives.

Ex: Compute $\int_0^5 12x^2y^3\,dx$ and $\int_0^1 12x^2y^3\,dy$ and the associated indefinite integrals.

First

$$\int 12x^2y^3\,dx = 4x^3y^3 + g(y)$$

Notice that the "constant" term in this indefinite integral is a function of y because "x sees y" as a constant.

$$\int_0^5 12x^2y^3\,dx = 4x^3y^3\Big|_0^5 = 4\cdot 5^3\cdot y^3 = 500y^3$$
$$\int 12x^2y^3\,dy = 3x^2y^4 + h(x)$$
$$\int_0^1 12x^2y^3\,dy = 3x^2y^4\Big|_0^1 = 3x^2\cdot 1^4 - 0 = 3x^2$$

Ex: Compute

$$\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx \qquad \int_0^4\int_0^2 y^3e^{2x}\,dx\,dy$$

$$\begin{aligned}\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx &= \int_0^2 \frac{1}{4}y^3e^{2x}\Big|_0^4\,dx\\ &= \int_0^2 64e^{2x}\,dx\\ &= 32e^{2x}\Big|_0^2 = 32(e^4-1)\end{aligned}$$

$$\begin{aligned}\int_0^4\int_0^2 y^3e^{2x}\,dx\,dy &= \int_0^4 \frac{1}{2}y^3e^{2x}\Big|_0^2\,dy\\ &= \int_0^4 \frac{1}{2}(e^4-1)y^3\,dy\\ &= \frac{1}{8}(e^4-1)y^3\Big|_0^4\\ &= \frac{1}{8}(e^4-1)\cdot 256 = 32(e^4-1)\end{aligned}$$

These integrals being equal is no coincidence:

Theorem 2. *Fubini's Theorem If f is continuous on $R=[a,b]\times[c,d]$ then*

$$\int\int_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

(There are more general conditions f could satisfy: f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the integral exists.)

Ex: Compute

$$\int\int_R ye^{-xy}\,dA$$

where $R = [0,2] \times [0,3]$.

Since this looks annoying to integrate y first, let's integrate w.r.t. x first. Then

$$\begin{aligned}\int\int_R ye^{-xy}\,dA &= \int_0^3\int_0^2 ye^{-xy}\,dx\,dy = \int_0^3 y\left(\frac{-1}{y}e^{-xy}\right)\Big|_0^2 dy \\ &= \left(y+\frac{1}{2}e^{-2y}\right)\Big|_0^3 = \left(3+\frac{1}{2}e^{-6}\right)-\left(0+\frac{1}{2}\right) \\ &= \frac{5}{2}+\frac{1}{2}e^{-6}\end{aligned}$$

Let's end with a volume example:

Ex: Find the volume of the solid bounded by the surface $z = 1 + e^x \sin y$ and the planes $x = 1$, $x = -1$, $y = 0$, $y = \pi$, and $z = 0$.

Since $e^x > 0$ and $\sin y \geq 0$ on $0 \leq y \leq \pi$, we have $z \geq 0$. The base of this solid is $R = [-1,1] \times [0,\pi]$ and its height is z. Thus,

$$\begin{aligned}\text{Vol} &= \int\int_R \left(1+e^x\sin y\right)\,dA = \int_{-1}^1\int_0^\pi \left(1+e^x\sin y\right)\,dy\,dx \\ &= \int_{-1}^1 \left(y-e^x\cos y\right)\Big|_0^\pi dx = \int_{-1}^1 \left[(\pi+e^x)-(-e^x)\right]\,dx \\ &= \int_{-1}^1 \left(2e^x+\pi\right)\,dx = \left(2e^x+\pi x\right)\Big|_{-1}^1 = (2e+\pi)-(2e^{-1}-\pi) \\ &= 2e-2e^{-1}+2\pi\end{aligned}$$

15.3 Double Integrals over General Regions

Let's consider the problem of integrating the function $z = 5$ over the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$

So, we're computing $\int\int_D 5\,dA$ where D is the triangular region. How do we compute this? Slices

Recall that to perform a double integral, we compute the inner integral first by holding one variable constant and integrating w.r.t. the other. Let's say we integrate w.r.t. x first. This means that we fix y and integrate from the smallest x-value to the largest at this y-value:

"horizontal slices"

So, the bounds on the x-integral are $0 \leq x \leq 1-y$. Now, all we have to do is add up over all the possible

y-values. So

$$\int_0^1 \int_0^{1-y} 5\,dx\,dy = \int 015x\Big|_0^{1-y} dy = \int_0^1 5 - 5y\,dy$$
$$= \left(5y - \frac{5}{2}y^2\right)\Big|_0^1 = 5 - \frac{5}{2}$$
$$= \frac{5}{2}$$

Comments:

1. The outside integral should NEVER have a variable in it!
2. Always sketch the region of integration. It really helps when setting up bounds.
3. To find bounds, first sketch the region, then decide which to take slices:
   - vertical (holding x constant first): look from the bottom to top:

     $$\int\int_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

   - horizontal (holding y constant first) look from left to right:

     $$\int\int_D f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

4. Sometimes integrating one way is easier than another

Ex: Compute $\int\int_D x\,dA$ where D is the region bounde by the parabolas $y = 1 - x^2$ and $y = x^2 - 1$

Step 1: Sketch the region!

Notice that vertical slices work better here since to do horizontal would require splitting the integral into two pieces (also the bounds wouldn't be as nice).

Step 2: Set up the integral

$$\begin{aligned}\int\int_D x\,dA &= \int_{-1}^{1}\int_{x^2-1}^{1-x^2} x\,dy\,dx = \int_{-1}^{1} xy\Big|_{x^2-1}^{1-x^2}\,dx\\ &= \int_{-1}^{1} x\left[(1-x^2)-(x^2-1)\right]\,dx = \int_{-1}^{1} 2x-2x^3\,dx\\ &= \left(x^2-\frac{1}{2}x^4\right)\Big|_{-1}^{1} = \left(1-\frac{1}{2}\right)-\left(1-\frac{1}{2}\right)=0\end{aligned}$$

From now on, we'll focus more on setting up integrals.

Ex: Set up the integral to find the area of the region bounded by $x=y^2-1$ and $y=x-1$

Notice that the volume of an object with height 1 is equal to the area of its base (ignoring units, of course). So, area of $D = A(D) = \int\int_D 1\,dA = \int\int_D dA$

First, sketch the region:

Horizontal slices will work more nicely this time. The left function is $x=y^2-1$ and the right function is $x=y+1$. Now, we just need the bounds on y. We find the max and min values of y. Plugging $x=y^2-1$ into $y=x-1$ gives

$$y=y^2-2 \quad\Longrightarrow\quad y^2-y-2=(y-2)(y+1)=0 \quad\Longrightarrow\quad y=-1,2$$

Thus:

$$\text{Area of } D = A(D) = \int\int_D dA = \int_{-1}^{2}\int_{y^2-1}^{y+1} dA$$

You also need to be able to read the region of integration off of a double integral:

Ex: Compute $\int_0^2 \int_{y^2}^4 y\cos(x^2)\,dx\,dy$.

As it stands we cannot compute it since we cannot compute $\int \cos(x^2)\,dx$... However, we can try to switch the order in which we integrate to $dy\,dx$. To do this, sketch the region:

- The outer integral says
$$0 \le y \le 2$$
- x goes between y^2 and 4 for any fixed y, so
$$y^2 \le x \le 4 \,.$$

We can then draw the region

So, rewriting the integral we have

$$\begin{aligned}
\int_0^2 \int_{y^2}^4 y\cos(x^2)\,dx\,dy &= \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx \\
&= \int_0^4 \frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{x}}\,dx = \int_0^4 \frac{1}{2}x\cos(x^2)\,dx \\
&= \int_0^{16} \frac{1}{4}\cos(u)\,du = \frac{1}{4}\sin u\Big|_0^{16} = \frac{1}{4}\sin 16
\end{aligned}$$

Last comment: If $D = D_1 \cup D_2$, then

$$\int\int_D f(x,y)\,dA = \int\int_{D_1} f(x,y)\,dA + \int\int_{D_2} f(x,y)\,dA$$

that is

Sometimes there's no choice but to split the integral in pieces.

15.4 Double Integrals in Polar Coordinates

Consider the integral $\int\int_D e^{x^2+y^2}\,dA$ where D is the unit disk. How can we compute it? The answer is polar coordinates. Let's practice describing regions in polar coordinates.
Ex: Describe the following regions in polar coordinates

1. $D = \{(r,\theta) | r \le 1\} = \{(r,\theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

2. $D = \{(r,\theta) | 1 \le r \le 2, 0 \le \theta \le \pi\}$

3. $D = \{(r,\theta) | r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$

These types of regions are called polar rectangles since if you graph them in the r,θ-plane, they're rectangles. The most general polar rectangle is a sector of the form

$$D = \{(r,\theta) | r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}, \qquad (0 \le \theta_2 \le \theta_1 \le 2\pi)$$

The area of D is

$$\begin{aligned} A &= \frac{1}{2} r_2^2 \Delta\theta - \frac{1}{2} r_1^2 \Delta\theta = \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta \\ &= r^* \Delta r \Delta\theta \end{aligned}$$

where $r^* = \frac{1}{2}(r_1 + r_2)$. This tells us $dA = r\,dr\,d\theta$. This means we can change from Cartesian to polar, we have

$$\int\int_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Ex: Compute $\int\int_D e^{x^2+y^2}\,dA$ where D is the unit disk

D in polar coordinates is $\{(r,\theta) | r \le 1\}$. The integrand becomes e^{r^2} because $r^2 = x^2 + y^2$, so we get

$$\begin{aligned} \int\int_D e^{x^2+y^2}\,dA &= \int_0^{2\pi} \int_0^1 e^{r^2} r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2} e^u\,du\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(e-1)\,d\theta = \pi(e-1) \end{aligned}$$

Regions of integration need not be polar rectangles. Consider the following problem from Calc II:

Ex: Find the area enclosed by one petal of the rose

$$r = \cos 3\theta$$

The rose looks like

We know $\cos 3\theta = 0$ when $3\theta = \frac{\pi}{2} + n\pi$ iff $\theta = \frac{\pi}{6} + \frac{n\pi}{3}$. Taking $\frac{-\pi}{6} \leq \theta \leq \frac{\pi}{6}$, we get the indicated petal. So to get the bounds on r we fix a θ-value (a ray coming out of the origin) and find an "inner" and "outer" bound on r. In this case, $0 \leq r \leq \cos 3\theta$. So

$$\begin{aligned}
A(D) &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4}(1+\cos 6\theta)\,d\theta \\
&= \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{1}{4}\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin\pi\right) - \left(\frac{-\pi}{6} + \frac{1}{6}\sin(-\pi)\right)\right] \\
&= \frac{\pi}{12}
\end{aligned}$$

Ex: Set up an integral giving the volume of the region bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$.

First, we rewrite these in polar coordinates paraboloid: $z = 4 - r^2$, cylinder: $r^2 = 2r\sin\theta$ iff $r = 2\sin\theta$.
In the cylinder, $r = 0$ iff $2\sin\theta = 0$ iff $\theta = \pi + n\pi$.
So, letting θ go from 0 to π, we get the cylinder. The bounds on r go from 0 to $2\sin\theta$. Thus

$$\begin{aligned}
\text{Vol} &= \int\int_D z\,dA = \int_0^{\pi}\int_0^{2\sin\theta} (4-r^2)r\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta} (4r - r^3)\,dr\,d\theta \\
&= \int_0^{\pi} (8\sin^2\theta - 4\sin^4\theta)\,d\theta = \int_0^{\pi}\left(\frac{5}{2} - 2\cos 4\theta\right)d\theta = \int_0^{\pi}\left(\frac{5}{2} - 2\cos 2\theta - \frac{1}{2}\cos 4\theta\right)d\theta \\
&= 10\pi
\end{aligned}$$

15.7 Triple Integrals

As you might imagine, triple integrals are defined using a triple Riemann sum. We'll leave the details of that to the book.
Let's start with the most basic example:
Ex: Compute the triple integral of $f(x, y, z) = x^2 y e^{xyz}$ over the box $B = [0,1] \times [1,2] \times [2,3]$.

$$
\begin{aligned}
\iiint_B x^2 y e^{xyz}\, dV &= \int_0^1 \int_1^2 \int_2^3 x^2 y e^{xyz}\, dz\, dy\, dx \\
&= \int_0^1 \int_1^2 x e^{xyz}\Big|_2^3\, dy\, dx = \int_0^1 \int_1^2 \left(x e^{3xy} - x e^{2xy}\right) dy\, dx \\
&= \int_0^1 \left(\frac{1}{3}e^{3xy} - \frac{1}{2}e^{2xy}\right)\Big|_1^2\, dx = \int_0^1 \left[\left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x}\right) - \left(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\right)\right] dx \\
&= \int_0^1 \left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x} - \frac{1}{3}e^{3x} + \frac{1}{2}e^{2x}\right) dx \\
&= \left(\frac{1}{18}e^{6x} - \frac{1}{8}e^{4x} - \frac{1}{9}e^{3x} + \frac{1}{4}e^{2x}\right)\Big|_0^1 \\
&= \frac{e^6}{18} - \frac{e^4}{8} - \frac{e^3}{9} + \frac{e^2}{4} - \left(\frac{1}{18} - \frac{1}{8} - \frac{1}{9} + \frac{1}{4}\right)
\end{aligned}
$$

There is, of course, no reason to stick to boxes.
Let's say our region is E. Let's orient the axes as such:

then when setting up bounds, they look as follows:
x: "back to front"
y: "left to right"
z: "bottom to top"
Again, sketching the region will be important! Now, once we've figured out the bounds on the inside integral, the outer two integrals' bounds come from setting up a double integral over a "shadow" region: If the inside integral is respect to $\begin{matrix} x \\ y \\ z \end{matrix}$, then we look at the shadow of E in the $\begin{matrix} yz \\ xy \\ xy \end{matrix}$-plane and set up the double integral over that.
Let's make this concrete with an example.
Ex: Set up the integral to compute the volue of E, where E is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 2$, and $x + y + z = 4$.

Begin by sketching

Now, we need an order of integration. Let's integrate y first... because, why not?
So we look from left to right and see that the left function is $y = 0$ and the right function is $x + y + z = 4 \iff y = 4 - x - z$. So the inside integral is $\int_0^{4-x-z} \, d dV$. The shadow E makes in the xz-plane is

So, the volume is:

$$\text{Vol}(E) = \int\int\int_E dV = \int_0^2 \int_0^{4-x} \int_0^{4-x-z} y \, d \, dz \, dx$$

As with double integrals, we may need to switch the order of integration.
Ex: Rewrite $\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz \, dz \, dx \, dy$ using $dy \, dz \, dx$.

We begin by sketching the region

It's simple to see here that y goes from 0 to 4. Now, the shadow in the xz-plane is:

So,

$$\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz\,dz\,dx\,dy = \int_{-1}^1 \int_{2x^2}^{x^2} \int_0^4 xyz\,dy\,dz\,dx$$

Now for an application let's take a detour.

15.5 Mass and Center of Mass

Recall that the center of mass of a system is the point, if all the mass of the system were concentrated there, the total moment is the same as the original.

15.5.1 Discrete Case

In the discrete case of masses m_i at points (x_i, y_i), the total moment or moment of mass about the

1. y-axis is $M_y = \sum_i m_i x_i$

2. x-axis is $M_x = \sum_i m_i y_i$

So, the center of mass has coordinates $(\overline{x}, \overline{y})$ where

$$\left(\sum_i m_i\right)\overline{x} = \sum_i m_i x_i \qquad \left(\sum_i m_i\right)\overline{y} = \sum_i m_i y_i$$

Thus

$$\overline{x} = \frac{M_y}{\text{total mass}} \qquad \overline{y} = \frac{M_x}{\text{total mass}}$$

We can do this for a system in $\mathbb{R}^3$ as well,

$$\overline{z} = \frac{M_{xy}}{\text{total mass}}$$

where $M_{xy} = \sum_i m_i z_i$ is the total moment about the xy-plane.

15.5.2 Continuous Case

Let's suppose we had a lamina D in $\mathbb{R}^2$ with density function $\rho(x, y)$. Then the total mass of D is

$$\boxed{\text{mass} = m = \int\int_D \rho(x,y)\,dA}$$

Using the discrete case as a guide, we find that

$$M_y = \int\int_D x\rho(x,y)\,dA \qquad M_x = \int\int_D y\rho(x,y)\,dA$$

So the center of mass of D has coordinates

$$\boxed{(\overline{x}, \overline{y}) = \left(\frac{\int\int_D x\rho\,dA}{\int\int \rho\,dA}, \frac{\int\int_D y\rho\,dA}{\int\int_D \rho\,dA}\right)}$$

The equations are similar for solid regions in $\mathbb{R}^3$.

15.8 Cylindrical Coordinates

Ex: Compute the volume of the solid bounded by $z = x^2 + y^2$ and $z = 4$.

We start by sketching the region:

It looks easiest to start by integrating z first

$$\text{Vol} = \int_?^? \int_?^? \int_{x^2+y^2}^4 dz\, d?\, d?$$

The shadow of this region in the xy-plane is just the disk of radius 2:

We could set up bounds in cartesian here, but since this is a disk, polar coordinates are easiest. The function we're integrating over the disk is

$$f(x, y) = \int_{x^2+y^2}^4 dz \iff f(r\cos\theta, r\sin\theta) = \int_{r^2}^4 dz$$

So we should have

$$\begin{aligned}
\text{Vol} &= \int_0^{2\pi} \int_0^2 \left(\int_{r^2}^4 dz \right) dr\, d\theta = \int_0^{2\pi} \int_0^2 z\Big|_{r^2}^4 r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \left(4 - r - r^3\right) dr\, d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2 d\theta \\
&= \int_0^{2\pi} (8-4)\, d\theta = 8\pi
\end{aligned}$$

Notice how this integral was done in polar coordinates, but with z just tacked on. This is exactly what cylindrical coordinates are:

$$(r, \theta, z) \qquad \text{where} \qquad x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

As z is just tacked on, we find

$$\boxed{dV = r\, dr\, d\theta\, dz}$$

Cylindrical coordinates can help with triple integrals as polar did with double, as the previous example shows.

Ex: a) Write the point with cyindrical coordinates $\left(4, \frac{\pi}{3}, -2\right)$ in cartesian coordinates.
b) Write the point with cartesian coordinates $(-2, 2\sqrt{3}, 3)$ in cylindrical coordinates.

a)
$$(x, y, z) = (r\cos\theta, r\sin\theta, z) = \left(4\cos\frac{\pi}{3}, 4\sin\frac{\pi}{3}, -2\right) = (2, 2\sqrt{3}, -2)$$

b) $r^2 = x^2 + y^2 = 4 + 12 = 16$ implies $r = 4$

$$\tan\theta = \frac{y}{x} = -\sqrt{3} \qquad \Longleftrightarrow \qquad \theta = \frac{2\pi}{3} + n\pi$$

Take $\theta = \frac{2\pi}{3}$ since $(-2, 2\sqrt{3})$ is in quadrant II. So

$$(r, \theta, z) = \left(4, \frac{2\pi}{3}, 3\right)$$

15.9 Spherical Coordinates

As we can use cylinders to give coordinates on $\mathbb{R}^3$ we can also use spheres. These coordinates are obtained by rotating polar coordinates into $\mathbb{R}^3$. Spherical coordinates are (ρ, θ, ϕ) where ρ is the distance from the origin, θ is the angle made with the positive x-axis in the xy-plane, and ϕ is the angle made with the positive z-axis. So, we have

$$\rho \geq 0, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi\ .$$

The relation to cartesian is

$$\boxed{x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\theta}$$

We also have

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Ex: a) Write the point with spherical coordinates $\left(3, \frac{\pi}{2}, \frac{3\pi}{4}\right)$ in cartesian coordinates.
b) Write the point with cartesian coordinates $(-1, 1, -\sqrt{2})$ in spherical coordinates.

a)

$$x = \rho\cos\theta\sin\phi = 3\cos\frac{\pi}{2}\sin\frac{3\pi}{4} = 3\cdot 0\cdot\sqrt{2}2 = 0$$
$$y = \rho\sin\theta\sin\phi = 3\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 3\cdot 1\cdot\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$
$$z = \rho\cos\phi = 3\cos\frac{3\pi}{4} = \frac{-3\sqrt{2}}{2}$$

So $(x, y, z) = \left(0, \frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$

b)

$$\rho^2 = x^2 + y^2 + z^2 = 1 + 1 + 2 = 4 \qquad \Longrightarrow \rho = 2$$
$$z = \rho\cos\phi \Longleftrightarrow -2\sqrt{2} = 2\cos\phi \Longrightarrow \cos\phi = -\frac{\sqrt{2}}{2} \Longrightarrow \phi = \frac{3\pi}{4}$$
$$y = \rho\sin\theta\sin\phi \Longleftrightarrow 2\sin\theta\sin\frac{3\pi}{4} = \sqrt{2}\sin\theta \Longrightarrow \sin\theta = \frac{\sqrt{2}}{2} \Longrightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

Checking with x:

$$x = \rho\cos\theta\sin\phi \Longleftrightarrow -1 = 2\cos\theta\sin\frac{3\pi}{4} = \sqrt{2}\cos\theta$$

So $\cos\theta \le 0$ implies $\theta = \frac{3\pi}{4}$

Thus, $(\rho, \theta, \phi) = \left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$

In spherical coordinates,

$$\boxed{dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

Ex: Find the volume of the region inside the sphere $x^2+y^2+z^2 = 4z$ and above the cone $z = \sqrt{\frac{1}{3}(x^2+y^2)}$.

Let’s rewrite the equations in spherical coords

$$\rho^2 = x^2 + y^2 + z^2 = 4z = 4\rho\cos\phi$$

So $\phi = 4\cos\phi$

$$\begin{aligned}\rho\cos\phi = z = \sqrt{\frac{1}{3}\left(x^2+y^2\right)} = \sqrt{\frac{1}{3}\left(\rho^2\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right)} = \sqrt{\frac{1}{3}\rho^2\sin^2\phi} \\ = \frac{1}{\sqrt{3}}\rho\sin\phi \Longrightarrow \cos\phi = \frac{1}{\sqrt{3}}\sin\phi \Longrightarrow \tan\phi = \sqrt{3} \Longrightarrow \phi = \frac{\pi}{3}\end{aligned}$$

Rewriting the sphere in standard form gives

$$x^2 + y^2 + (z-2)^2 = 4$$

a sphere of radius 2 centered at $(0,0,2)$.
The cone is given by $\phi = \frac{\pi}{3}$, so a sketch of the region is

We end up with a "snow cone" type object. Then the volume is

$$\begin{aligned}\mathrm{Vol} &= \int\int\int_E dV = \int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{4\cos\phi} \rho^2\sin\phi\, d\rho\, d\theta\, d\phi \\ &= \int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{64}{3}\cos^3\phi\sin\phi\, d\theta\, d\phi = \frac{128\pi}{3}\int_0^{\frac{\pi}{3}}\cos^3\phi\sin\phi\, d\phi \\ &= \frac{32\pi}{3}(-\cos^4\phi)\Big|_0^{\frac{\pi}{3}} = \frac{32\pi}{3}\left(-\frac{1}{16} - (-1)\right) = 10\pi\end{aligned}$$

15.10 Change of Variables

Ex: Compute $\int\int_D 3xy\,dA$ where D is the region bounde by $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$.

Sketch:

Notice that, to do this integral would require splitting the region into 3 pieces. There must be an easier way... Notice that the opposite sides of the parallelogram are described by the same function

$$x-2y=-4,0 \qquad\qquad x+y=1,4$$

If we write $u=x-2y$ and $v=x+y$, then the region is bounded by $u=-4$, $u=0$, $v=1$, $v=4$ in the uv-plane... much simpler! We need to replace x and y, so we solve for them in terms of u and v:

$$\begin{cases} u = x-2y \\ v = x+y \end{cases}$$

Using algebra we solve for x and y giving us $x=\frac{1}{3}(2v+u)$ and $y=\frac{1}{3}(v-u)$. Our integrand becomes

$$3xy = \frac{1}{3}(v-u)(2v+u) = \frac{1}{3}(2v^2-uv-u^2)$$

What about dA?

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the <u>Jacobian</u> of the transformation and is given by

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

So in this example

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} - \left(-\frac{2}{9}\right) = \frac{1}{3}$$

thus $dA=\frac{1}{3}\,du\,dv$. So, the integral becomes

$$\int\int_D 3xy\,dA = \int_1^4\int_{-4}^0 \frac{1}{9}\left(2v^2-uv-u^2\right) du\,dv = \frac{164}{9}$$

If we write $T(u,v) = (x(u,v), y(u,v))$ to represent the transformation, then

$$\frac{\partial(x,y)}{\partial(u,v)} = \det DT(u,v)$$

Definition 3. *$T(u,v) = (x(u,v), y(u,v))$ is C^1 if its components have continuous first partials.*

15.10.1 Change of Variables Formula (2 variables)

Suppose $T(u,v) = (x(u,v), y(u,v))$ is C^1 and sends the region S in the uv-plane to the region R in the xy-plane. If the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero at all points in S, $f(x,y)$ is continuous on R, and T is one-to-one on S, except maybe on the boundary of S, then

$$\begin{aligned} \int\int_R f(x,y)\,dA &= \int\int_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= \int\int_{T^{-1}(R)} f(T(u,v))|\det DT|\,du\,dv \end{aligned}$$

This theorem can also be used to make integrands simpler. This more like u-substitution. Before an example of this, a neat trick from linear algebra:

$$\det(A^{-1}) = \frac{1}{\det A}$$

If $T(u,v) = (x(u,v), y(u,v))$, then $T^{-1}(x,y) = (u(x,y), v(x,y))$. So, $\frac{\partial(u,v)}{\partial(x,y)} = \det D(T^{-1})$. But $D(T^{-1}) = DT^{-1}$. Thus

$$\frac{\partial(u,v)}{\partial(x,y)} = \det D(T^{-1}) = (\det DT)^{-1} = \frac{1}{\det DT} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

So

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

Ex: Compute $\int\int_R \cos\left(\frac{y-x}{y+x}\right)\,dA$ where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, $(0,1)$.

$\cos\left(\frac{y-x}{y+x}\right)$ looks pretty hard to integrate... If we write $u = y-x$ and $v = y+x$, then we get $\cos\left(\frac{y-x}{y+x}\right) = \cos\left(\frac{u}{v}\right)$, a bit better. What happens to R? More like, what maps to R? First, let's sketch R:

What is the region S which maps to R?

xy-plane	uv-plane
$x+y=2$	$v=2$
$x+y=1$	$v=1$
$x=0$	$\begin{cases} u=y-x=y \\ v=y+x=y \end{cases} \Longrightarrow u=v$
$y=0$	$\begin{cases} u=y-x=-x \\ v=y+x=x \end{cases} \Longrightarrow u=-v$

So, S is bounded by $v=2$, $v=1$, $u=v$, and $u=-v$. Pictorially:

Now, we need the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1=-2 \quad\Longrightarrow\quad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

Thus,

$$\begin{aligned}\int\int_R \cos\left(\frac{y-x}{y+x}\right)\,dA &= \int_1^2\int_{-v}^{v}\cos\left(\frac{u}{v}\right)\left|\frac{-1}{2}\right|\,du\,dv \\ &= \frac{1}{2}\int_1^2\int_{-v}^{v}\cos\left(\frac{u}{v}\right)\,du\,dv = \frac{3}{2}\sin(1)\end{aligned}$$

There is a corresponding 3-variable version of the theorem. If $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ and $T(S) = R$ then

$$\int\int\int_R f(x,y,z)\,dV = \int\int\int_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$