Example 1. *Evaluate the integrals:*

1.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$$

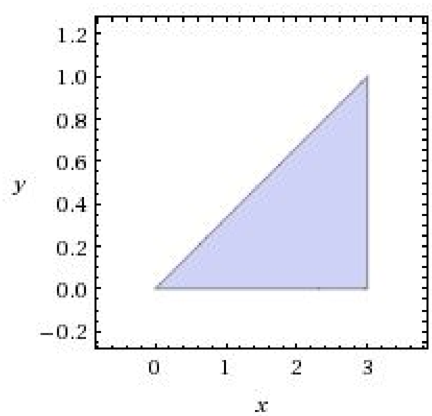
2.

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$$

3.

$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2(x)}\,dx\,dy\,.$$

1. First we sketch:



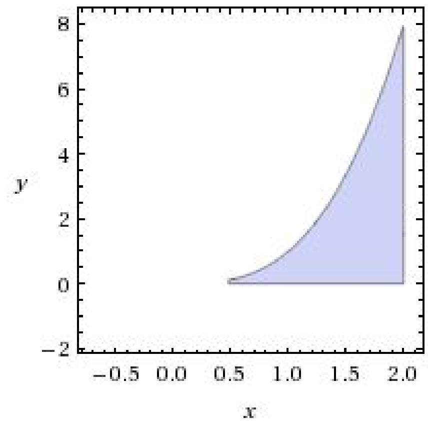
Let's reorder the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx\,.$$

Now let's compute the later:

$$\begin{aligned}\int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx &= \int_0^3 ye^{x^2}\Big|_{y=0}^{\frac{1}{3}x}\,dx \\ &= \int_0^3 \frac{1}{3}xe^{x^2}\,dx \\ &= \frac{1}{6}e^{x^2}\Big|_{x=0}^{3} \\ &= \frac{1}{6}\left(e^9-1\right)\end{aligned}$$

2. First we sketch



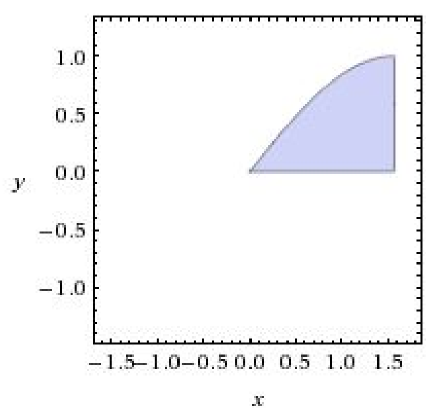
Let's reorder the integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy = \int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx\,.$$

Now,

$$\begin{aligned}\int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx &= \int_0^2 y e^{x^4}\Big|_{y=0}^{x^3} dx \\ &= \int_0^2 x^3 e^{x^4}\, dx \\ &= \frac{1}{4} e^{x^4}\Big|_{x=0}^{2} \\ &= \frac{1}{4}\left(e^{16}-1\right)\end{aligned}$$

3. First we sketch



Let's reorder the integral

$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2(x)}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2(x)}\, dy\, dx$$

Now,

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^{\sin(x)}\cos(x)\sqrt{1+\cos^2(x)}\,dy\,dx &= \int_0^{\frac{\pi}{2}} y\cos(x)\sqrt{1+\cos^2(x)}\Big|_0^{\sin(x)}\,dx\\ &= \int_0^{\frac{\pi}{2}}\sin(x)\cos(x)\sqrt{1+\cos^2(x)}\,dx\\ &= \int_1^2 \frac{1}{2}\sqrt{u}\,du\\ &= \frac{1}{3}u^{\frac{3}{2}}\Big|_1^2\\ &= \frac{1}{3}\left(2^{\frac{3}{2}}-1\right)\end{aligned}$$