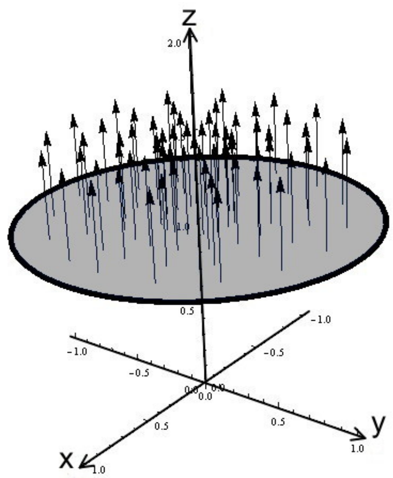
1. (8pts) Let C be the curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, 1 \rangle$. Stokes' theorem tells us that for a differentiable vector field $\mathbf{F}$

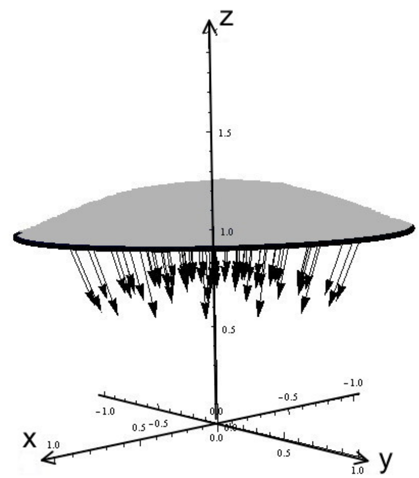
$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \bullet d\mathbf{S}$$

for any oriented smooth surface S. The surfaces below are possible choices for S where the vectors indicate the direction of the normal field. Which of the following is an appropriate choice for S? The curve C is drawn in bold in each picture and the positive directions on the axes are labeled. *Hint:* Be mindful of the orientation of C!

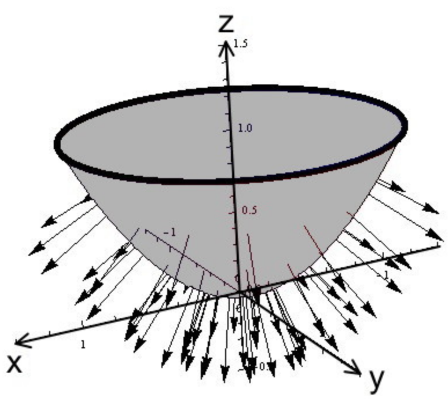
(a)



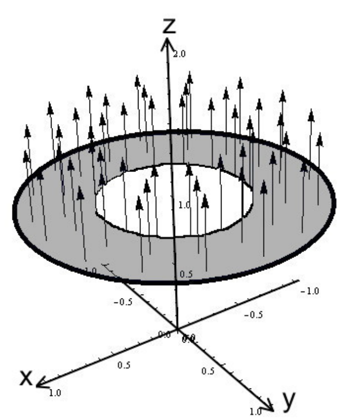
(b)



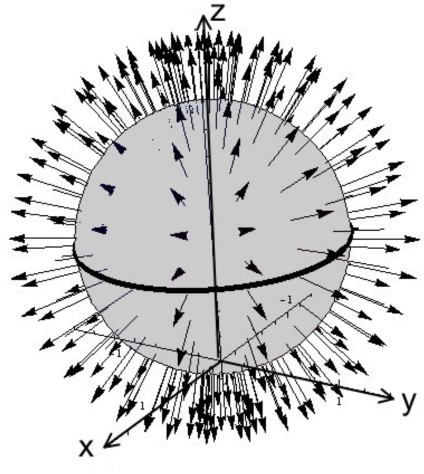
(c)



(d)



(e)



Solution. This question boils down to figuring out whether the given surface has C as its boundary and has orientation consistent with the orientation on C. Since the annulus has both C and the inner circle as its boundary, it is immediately out. The sphere has no boundary, so that is out. Now, for a surface to have orientation consistent with the orientation of its boundary, if you walk around the boundary in the direction of its orientation with your head pointing in the direction of the normal vector field, then the surface will be on your left. This leaves the disk with upward normals as the correct answer.

2.(8pts) Let T be the surface $\mathbf{r}(u,\theta) = \big\langle u, u^2\cos(\theta), u^2\sin(\theta)\big\rangle$ for $0 \leqslant u \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. This is the surface you get by rotating $y = x^2$, $0 \leqslant x \leqslant 1$, around the y axis. If the density at a point on this surface is given by $24x^5 + 4x^3$, find the mass.
Remark: The density has been chosen so that a simple substitution can be used to evaluate the integral you should get.

(a) $\dfrac{4\pi}{3}5^{\frac{3}{2}}$ (b) 20π (c) $\pi\big(\sqrt{24}-2\big)$ (d) $\pi\sqrt{28}$ (e) $\pi(24)^{\frac{3}{2}}$

Solution.

$$\mathbf{r}_u = \langle 1, 2u\cos(\theta), 2u\sin(\theta)\rangle$$
$$\mathbf{r}_\theta = \big\langle 0, -u^2\sin(\theta), u^2\cos(\theta)\big\rangle$$
$$\mathbf{r}_u \times \mathbf{r}_\theta = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u\cos(\theta) & 2u\sin(\theta) \\ 0 & -u^2\sin(\theta) & u^2\cos(\theta) \end{vmatrix} = \big\langle 2u^3, -u^2\cos(\theta), -u^2\sin(\theta)\big\rangle$$
$$|\mathbf{r}_u \times \mathbf{r}_\theta| = \sqrt{4u^6 + u^4}$$

Hence

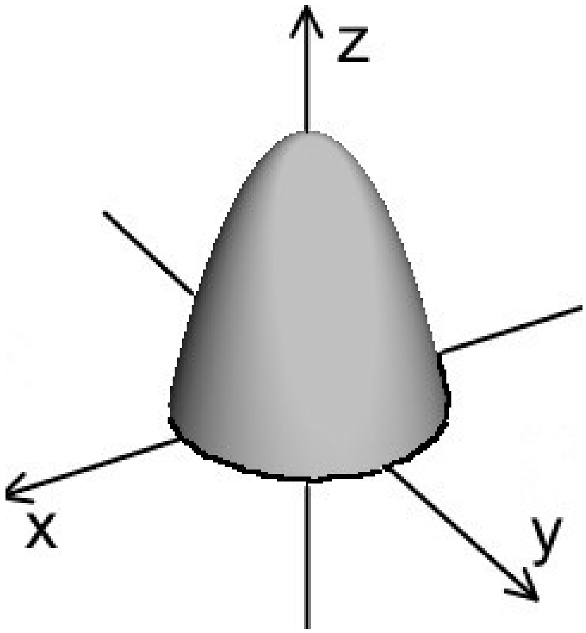
$$\text{Mass} = \iint\limits_T (24x^5 + 4x^3)\,dS = \iint\limits_{\substack{0\leqslant u\leqslant 1\\ 0\leqslant\theta\leqslant 2\pi}} (24u^5 + 4u^3)\sqrt{4u^6+u^4}\,dA =$$
$$\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (24u^5 + 4u^3)\sqrt{4u^6+u^4}\,du\right)$$

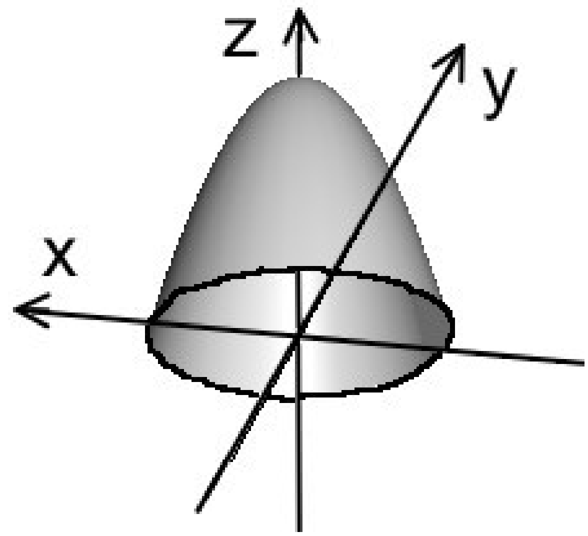
To do $\displaystyle\int_0^1 (24u^5+4u^3)\sqrt{4u^6+u^4}\,du$ use the substitution $v = 4u^6+u^4,\ dv = (24u^5+4u^3)\,du$.
Now isn't that handy.

$$\int_0^1 (24u^5 + 4u^3)\sqrt{4u^6+u^4}\,du = \int_0^5 \sqrt{v}\,dv = \frac{3}{2}v^{\frac{2}{3}}\Big|_0^5 = \frac{2}{3}5^{\frac{3}{2}}$$

Hence the mass is $2\pi\frac{2}{3}5^{\frac{3}{2}} = \frac{4}{3}\pi 5^{\frac{3}{2}}$.

3.(8pts) Find $\iint_T \mathbf{F} \bullet d\mathbf{S}$ where $\mathbf{F} = \left\langle x + ye^{z^2} + ze^{y^2}, y + xe^{z^2} + ze^{x^2}, z \right\rangle$ and where T is the surface in the pictures. (The two pictures are two views of the same surface.) The boundary of T is the unit circle in the xy plane. Let D be the unit disk in the xy plane. Then $T \cup D$ is the boundary of a solid E whose volume is 2π. The orientation on T is outward with respect to E. You are looking at the picture from underneath. This problem will require some thought since all you know about T and E are what you see in the picture plus the volume of E.





(a) 6π (b) -6π (c) 2π (d) -2π (e) $\pi(e^2 - 2e)$

Solution. $\operatorname{div} F = 3$ so $\iiint_E \operatorname{div} F \, dV = 6\pi$. Hence

$$\iint_{\partial E} \mathbf{F} \bullet d\mathbf{S} = \iint_{T \cup D} \mathbf{F} \bullet d\mathbf{S} = \iint_T \mathbf{F} \bullet d\mathbf{S} + \iint_D \mathbf{F} \bullet d\mathbf{S} = 6\pi$$

T is oriented the way we want and D is oriented so that the normal vector points down: the unit normal is $\langle 0, 0, -1 \rangle$ and the field restricted to D is $\langle x + y, y + x, 0 \rangle$ since this is what we get when $z = 0$. Then

$$\iint_D \langle x + y, y + x, 0 \rangle \bullet \langle 0, 0, -1 \rangle \, dS = \iint_{x^2+y^2 \leqslant 1} 0 \, dS = 0$$

Hence $\iint_T \mathbf{F} \bullet d\mathbf{S} = 6\pi$.

4.(7pts) Determine the surface area of the part of $z = xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

(a) $\frac{2\pi}{3}(2^{\frac{3}{2}} - 1)$

(b) $\frac{2\pi}{3}(2^{\frac{2}{3}} - 1)$

(c) $\frac{2\pi}{3}(2^{\frac{3}{2}} + 1)$

(d) $\frac{2\pi}{3}(2^{\frac{2}{3}} + 1)$

(e) 2π

Solution. The partial derivatives are given by $f_x = y$ and $f_y = x$. The surface area is given by $S = \iint_D \sqrt{x^2 + y^2 + 1}\, dA$. After converting to polar coordinates we obtain

$$\int_0^{2\pi} \int_0^1 r\sqrt{r^2 + 1}\, dr\, d\theta = \int_0^{2\pi} \frac{1}{3}(2^{\frac{3}{2}} - 1)\, d\theta = \frac{2\pi}{3}(2^{\frac{3}{2}} - 1).$$

5.(7pts) We are asked to find absolute maximum and minimum values of $f(x,y,z)$ with respect to the constraints $g(x,y,z) = 10$ and $h(x,y,z) = e$. We are given that the five points $(1,1,1)$, $(0,1,1)$, $(0,1,-1)$, $(1,-1,1)$ and $(-1,1,-1)$ satisfy both constraint equations. Each of the five points satisfies an additional condition:

$$\nabla f(1,1,1) = -2\nabla g(1,1,1) + 3\nabla h(1,1,1)$$
$$\nabla f(0,1,1) = 6\nabla g(0,1,1)$$
$$\nabla f(0,1,-1) \bullet \big((\nabla g(0,1,-1) \times \nabla h(0,1,-1)\big) = 0$$
$$\nabla f(-1,1,-1) \bullet \big((\nabla g(-1,1,-1) \times \nabla h(-1,1,-1)\big) \neq 0$$
$$\nabla f(1,-1,1) = \mathbf{0}$$

Which point can **not** be an absolute extremum?

(a) $(-1,1,-1)$. (b) $(1,-1,1)$. (c) $(1,1,1)$. (d) $(0,1,1)$. (e) $(0,1,-1)$.

Solution. The points where a possible absolute maximum and minimum can occur are points on the two constraints such that ∇f is in the plane formed by ∇g and ∇h. So ∇f is a linear combination of ∇g and ∇h $\nabla f \bullet \big(\nabla g \times \nabla h\big) = 0$, scalar triple product of ∇f, ∇g, ∇h is zero, or the hessian of $\nabla f, \nabla g$, ∇h is zero. All those occur expect for the point $(-1,1,-1)$.

6.(7pts) Which of the following is a **minimum** of $f(x, y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$?

(a) $(-1, 0)$

(b) $(0, 1)$

(c) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(d) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

(e) None of these points is a minimum.

Solution. Letting $g(x, y) = x^2 + y^2$, the equations to solve are:

$$\begin{cases} \nabla f & = \lambda \nabla g \\ g(x, y) & = 1 \end{cases}$$

which, written out are:

$$\begin{cases} 2x & = 2\lambda x \\ 4y & = 2\lambda y \\ x^2 + y^2 & = 1 \end{cases}$$

The first equation says that either $x = 0$ or $\lambda = 1$. If $x = 0$, then the third equation gives that $y = \pm 1$; so $(0, 1)$ and $(0, -1)$ are possible extrema. If $\lambda = 1$, then the second equation gives that $y = 0$ which, by the third equation, gives that $x = \pm 1$. This gives $(1, 0)$ and $(-1, 0)$ as two more possible extrema. This eliminates $(0, 0)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ as possible answers. Plugging in the possible extrema, we find that

$$f(1, 0) = f(-1, 0) = 1$$
$$f(0, 1) = f(0, -1) = 2$$

and so $(1, 0)$ and $(-1, 0)$ are the minima of f subject to $x^2 + y^2 = 1$. Thus $(-1, 0)$ is the correct answer.

7.(7pts) Let $f(x,y) = x^2 + 6xy - 3y^2$. Find and classify all critical points.

(a) $(0,0)$, saddle point.

(b) $(0,0)$ and $(1,1)$, both saddle points.

(c) $(0,0)$, local maximum.

(d) $(-1,1)$, local minima, $(0,0)$, local maximum.

(e) $(0,0)$, local maximum.

Solution. We first compute $f_x = 2x + 6y$, $f_y = 6x - 6y$, $f_{xx} = 2$, $f_{xy} = 6$, $f_{yy} = -6$. The critical points are the solutions to the equations

$$2x + 6y = 0$$
$$6x - 6y = 0$$

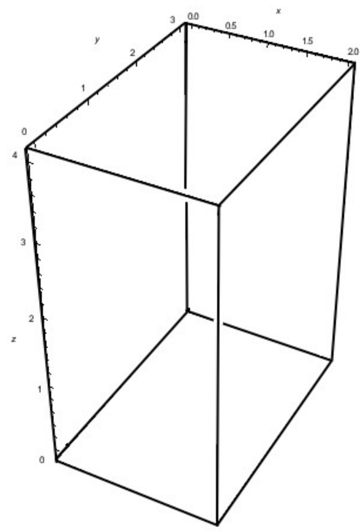
The only solutions is $x = y = 0$. Thus $(0,0)$ is the only critical points. The value of the Hessian -48 and we conclude that $(0,0)$ is a saddle point

8.(7pts) Find the curvature when $t = 0$ of $\mathbf{r}(t) = \left\langle \cos t, \sin t, e^t \right\rangle$.

(a) $\frac{\sqrt{3}}{2\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{\sqrt{2}}$ (e) $\frac{3}{2\sqrt{2}}$

Solution. $\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. $\mathbf{r}'(t) = \left\langle -\sin t, \cos t, e^t \right\rangle$ and $\mathbf{r}'(t) = \langle 0, 1, 1 \rangle$. $\mathbf{r}''(t) = \left\langle -\cos t, -\sin t, e^t \right\rangle$ and $\mathbf{r}''(t) = \langle -1, 0, 1 \rangle$. $\langle 0, 1, 1 \rangle \times \langle -1, 0, 1 \rangle = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, -1, 1 \rangle$.
$|\langle 1, -1, 1 \rangle| = \sqrt{3}$ and $|\langle 0, 1, 1 \rangle| = \sqrt{2}$. $\kappa(0) = \frac{\sqrt{3}}{2\sqrt{2}}$.

9.(7pts) Find $\displaystyle\iint_D \mathbf{F} \bullet d\mathbf{S}$ where $\mathbf{F} = \langle x^2, -2yz, z^2 \rangle$ and D is the surface of the cube E with vertices: $(0,0,0), (2,0,0), (2,3,0), (2,0,4), (0,3,0), (0,3,4), (0,0,4)$, and $(2,3,4)$, so $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 3$ and $0 \leqslant z \leqslant 4$.



(a) 48 (b) 36 (c) 24 (d) 0 (e) -24

Solution. Use the Divergence Theorem to converge this into a triple integral, $\displaystyle\iiint_E 2x\, dV$.

This becomes $\displaystyle\int_0^3 \int_0^4 \int_0^2 2x\, dx\, dz\, dy = 48.$

10.(7pts) Find the equation of the tangent plane to the surface $z = x + \ln(2x + y)$ at the point $(-1, 3, -1)$.

(a) $3x + y - z = 1$

(b) $\langle -1 + 3t, 3 + t, -1 - t\rangle$

(c) $-x + 3y - z = 11$

(d) $-x + y - 3z = 1$

(e) $3x - y + z = -7$

Solution. Let $f(x, y, z) = x + \ln 2x + y - z$. Then $\nabla f = \left\langle 1 + \frac{2}{2x + y}, \frac{1}{2x + y}, -1 \right\rangle$ so that $\nabla f(-1, 3, -1) = \langle 3, 1, -1\rangle$. Since $\nabla f(-1, 3, -1)$ defines a normal vector to the surface, an equation of the tangent plane is given by

$$\langle 3, 1, -1\rangle \bullet \langle x + 1, y - 3, z + 1\rangle = 0$$

or $3x + y - z = 1$.

11.(7pts) If $f(x,y) = \sin(xy)$ find the directional derivative $D_{\mathbf{u}}f$ at the point (x,y), where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(a) $\frac{x+y}{\sqrt{2}}\cos(xy)$ (b) $-\frac{x+y}{\sqrt{2}}\cos(xy)$ (c) $\frac{x+y}{\sqrt{2}}\sin(xy)$

(d) $-\frac{x+y}{\sqrt{2}}\sin(xy)$ (e) $\frac{x-y}{\sqrt{2}}\cos(xy)$

Solution. The gradient is given by $\nabla f = \langle y\cos(xy), x\cos(xy)\rangle$ so $D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} = \frac{x+y}{\sqrt{2}}\cos(xy)$.

12.(7pts) Find $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = \langle x, z, y \rangle$ and C is parametrized by $\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$, $0 \leqslant t \leqslant \dfrac{\pi}{2}$.

(a) $\dfrac{\pi}{2} - \dfrac{1}{2}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{1}{2}$ (d) $\pi - 2$ (e) $\pi + 2$

Solution. $\mathbf{F}(\mathbf{r}(t)) = \langle \cos t, t, \sin t \rangle$

$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = -\sin t \cos t + t \cos t + \sin t$

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_0^{\frac{\pi}{2}} (-\sin t \cos t + t \cos t + \sin t)\, dt$$

$$\int_0^{\frac{\pi}{2}} -\sin t \cos t\, dt = -\int_0^1 u\, du = \frac{1}{2} \text{ (where } u = \sin t)$$

$$\int_0^{\frac{\pi}{2}} t \cos t + \sin t\, dt = t \sin t \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Thus $\mathbf{F} \bullet d\mathbf{r} = -\dfrac{1}{2} + \dfrac{\pi}{2}$.

13.(7pts) Compute the flux $\iint_S \mathbf{F} \bullet d\mathbf{S}$, where $\mathbf{F} = \langle 4y^2, xy, xz \rangle$ and S is the surface $z = ye^x$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, with the upward orientation.

(a) $1 - e$ (b) $e - 1$ (c) e (d) $-e$ (e) $1 + 2e$

Solution. The parametrization is $\mathbf{r}(x, y) = \langle x, y, ye^x \rangle$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. The normal vector has positive z-coordinate so the orientation is up. Hence $\mathbf{F}\big(\mathbf{r}(x, y)\big) = \langle 4y^2, xy, xye^x \rangle$ and $d\mathbf{S} = \mathbf{r}_x \times \mathbf{r}_y \, dA = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & ye^x \\ 0 & 1 & e^x \end{vmatrix} dA = \langle -ye^x, -e^x, 1 \rangle \, dA.$

Hence

$$\text{Flux} = \iint\limits_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1}} \langle 4y^2, xy, xye^x \rangle \bullet \langle -ye^x, -e^x, 1 \rangle \, dA = \iint\limits_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1}} (-4y^3 e^x - xye^x + xye^x) \, dA =$$

$$-4 \int_0^1 \int_0^1 y^3 e^x \, dy \, dx = -4 \int_0^1 \left. \frac{y^4}{4} \right|_0^1 e^x \, dx = -\int_0^1 e^x \, dx = -e + 1$$

14.(7pts) Compute $\int \mathbf{x}(t)\,dt$ where $\mathbf{x}(t) = \sin(t)\mathbf{i} + e^t\mathbf{k}$.

(a) $(-\cos t + C_1)\mathbf{i} + C_2\mathbf{j} + (e^t + C_3)\mathbf{k}$

(b) $(-\cos t + C)\mathbf{i} + C\mathbf{j} + (e^t + C)\mathbf{k}$

(c) $(-\cos t + C_1)\mathbf{i} + (e^t + C_2)\mathbf{k}$

(d) $(-\cos t)\mathbf{i} + e^t\mathbf{k}$

(e) $-\cos t + e^t + C$

Solution.

$$\begin{aligned}\int \mathbf{x}(t)\,dt &= \int \left(\sin t)\mathbf{i} + e^t\mathbf{k}\right)\,dt \\ &= \int \left\langle \sin t, 0, e^t \right\rangle\,dt \\ &= \left\langle \int \sin t\,dt, \int 0\,dt, \int e^t\,dt \right\rangle \\ &= \left\langle -\cos t + C_1, C_2, e^t + C_3 \right\rangle = (-\cos t + C_1)\mathbf{i} + C_2\mathbf{j} + (e^t + C_3)\mathbf{k}\end{aligned}$$

15.(7pts) Find the center of mass of the thin plate D which has the shape of a half-disk consisting of the region below the **semicircle** bounded by $x^2 + y^2 = 9$ and above the x-axis. Assume that D has density $\rho(x, y) = \sqrt{x^2 + y^2}$ and mass 9π.

(a) $\left(0, \dfrac{9}{2\pi}\right)$ (b) $\left(0, \dfrac{3}{2\pi}\right)$ (c) $\left(0, \dfrac{2}{3\pi}\right)$ (d) $\left(0, \dfrac{2}{9\pi}\right)$ (e) $\left(0, \dfrac{\pi}{3}\right)$

Solution. The region and the density are symmetric about the y-axis, hence the x-coordinate of the center of mass is 0. To find the y-coordinate, we compute the moment about the x-axis. In polar coordinates,

$$M_x = \iint\limits_{x^2+y^2\leqslant 9} \sqrt{x^2+y^2}\, y\, dA = \int_0^{\pi}\int_0^{2} r(r\sin(\theta))r\, dr\, d\theta = \left(\int_0^{\pi} \sin(\theta)\, d\theta\right)\left(\int_0^{3} r^3\, dr\right) =$$

$$\left(-\cos\theta\Big|_0^{\pi}\right)\left(\frac{r^4}{4}\Big|_0^{3}\right) = 2\cdot\frac{81}{4} = \frac{81}{2}$$

The mass is

$$M = \iint\limits_{x^2+y^2\leqslant 9} \sqrt{x^2+y^2}\, dA = \int_0^{\pi}\int_0^{2} rr\, dr\, d\theta = \int_0^{\pi}\int_0^{3} r^2\, dr\, d\theta = 2\pi\,\frac{r^3}{3}\Big|_0^{3} = 9\pi$$

Hence,

$$\bar{y} = \frac{M_x}{\text{mass}(D)} = \frac{\frac{81}{2}}{9\pi} = \frac{9}{2\pi}$$

16.(7pts) Compute the curl of the vector field $\mathbf{F}(x,y,z) = \langle e^x \sin(2y), e^y \cos(2z), e^x \rangle$.

(a) $\langle 2e^y \sin(2z), -e^x, -2e^x \cos(2y) \rangle$

(b) $\langle -2e^y \sin(2z), e^x, 2e^x \cos(2y) \rangle$

(c) $2\,\langle e^y \sin(2z), -e^x, -e^x \cos(2y) \rangle$

(d) $\langle 2e^x \sin(2z), -e^x, -2e^x \cos(2y) \rangle$

(e) $\langle 2e^y \sin(2z), e^x, -2e^y \cos(2y) \rangle$

Solution.

$$\operatorname{curl} \mathbf{F} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \sin(2y) & e^y \cos(2z) & e^x \end{vmatrix} = \langle 2e^y \sin(2z), -e^x, 2e^x \cos(2y) \rangle$$

17.(7pts) Compute the angle between the vectors $\langle 1,1,0\rangle$ and $\langle 2,-1,-3\rangle$.

(a) $\arccos\left(\frac{1}{\sqrt{28}}\right)$ (b) π (c) $\pi/2$ (d) $\arccos\left(\frac{1}{28}\right)$

(e) $\pi/4$

Solution. The desired angle is given by

$$\theta = \arccos \frac{\langle 1,1,0\rangle}{\sqrt{2}} \bullet \frac{\langle 2,1,-3\rangle}{\sqrt{14}} = \arccos\left(\frac{1}{\sqrt{28}}\right)$$

where $\bullet$ stands for the dot product.

18.(7pts) Which of the following is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a} \bullet \mathbf{b} = 1$ and $\mathbf{a} \times \mathbf{b} \neq \langle 0, 0, 0 \rangle$.

(a) $(\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \times \mathbf{b})$ (b) $\mathbf{a} + \mathbf{b}$ (c) $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} \times \mathbf{b})$

(d) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}$ (e) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}$

Solution. We're looking for a vector which is parallel to $\mathbf{a} \times \mathbf{b}$, and it's obvious that $(\mathbf{a} \bullet \mathbf{b})(\mathbf{a} \times \mathbf{b})$ is parallel to $(\mathbf{a} \times \mathbf{b})$. (because $\mathbf{a} \bullet \mathbf{b} \neq 0$)

However, $\mathbf{a}+\mathbf{b}$, $(\mathbf{a}\times\mathbf{b})\times\mathbf{b}$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{a}$ are perpendicular to $\mathbf{a}\times\mathbf{b}$, and $(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}\times\mathbf{b})$ is not a vector.

19.(7pts) Find the *direction* of fastest increase of the function $f(x, y) = x^2 - y^3$ at the point $(2, 1)$.

(a) $\left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$ (b) $\langle 4, -1 \rangle$ (c) $\left\langle \frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$

(d) $\langle -3, 2, 1 \rangle$ (e) $\left\langle -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$

Solution. The gradient is $\nabla f = \left\langle 2x, -3y^2 \right\rangle$ which at $(2, 1)$ is $\langle 4, -3 \rangle$. The direction is $\left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$

20.(7pts) Define a transformation $x(u,w) = u^w$, $y(u,w) = w^u$ for $0 < u < \infty$, $0 < w < \infty$. Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,w)}$ of this transformation.

Recall: $\dfrac{d\,a^x}{dx} = a^x \ln(a)$.

(a) $u^w w^u\big(1 - \ln(u)\ln(w)\big)$ (b) $u^w \ln(u) + w^u \ln(w)$ (c) $(u+w)u^{w-1}w^{u-1}$

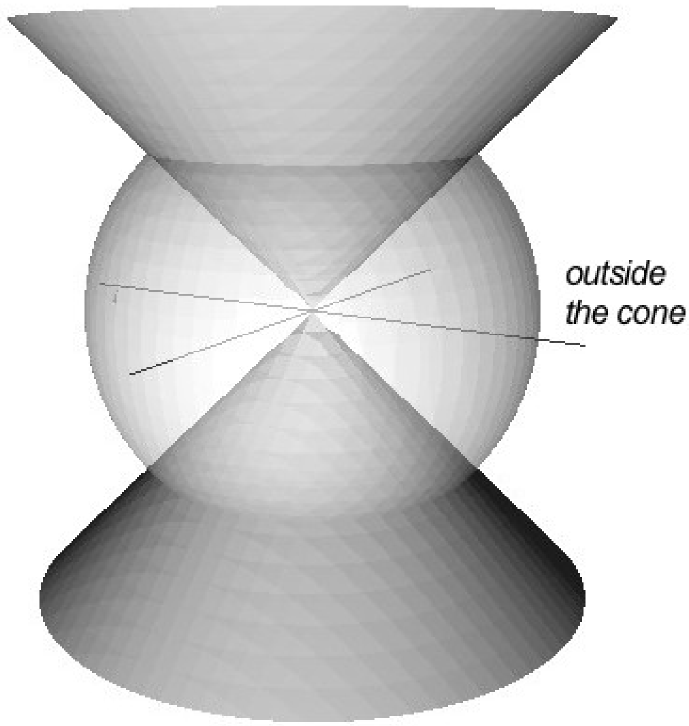
(d) $(u^2+w^2)u^{w-1}w^{u-1}$ (e) 0

Solution.

$$\frac{\partial(x,y)}{\partial(u,w)} = \det\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \det\begin{vmatrix} \dfrac{\partial u^w}{\partial u} & \dfrac{\partial u^w}{\partial w} \\ \dfrac{\partial w^u}{\partial u} & \dfrac{\partial w^u}{\partial w} \end{vmatrix} = \det\begin{vmatrix} wu^{w-1} & u^w \ln(u) \\ w^u \ln(w) & uw^{u-1} \end{vmatrix} =$$

$$u^w w^u - u^w w^u \ln(u)\ln(w) = u^w w^u\big(1 - \ln(u)\ln(w)\big)$$

21.(7pts) Find the volume that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cone $z^2 = x^2 + y^2$.



(a) $\frac{8}{3}\pi$ (b) $\frac{3}{8}\pi$ (c) π (d) 2π (e) $\frac{\pi}{2}$

Solution. In spherical coordinates the sphere becomes $\rho = \sqrt{2}$. To convert the cone into spherical coordinates we add z^2 to both sides of the equation defining the cone so that $2z^2 = x^2 + y^2 + z^2$ which becomes

$$2\rho^2 \cos^2 \phi = \rho^2$$

Therefore $\phi = \arccos\left(\frac{1}{\sqrt{2}}\right)$ which gives $\phi = \frac{\pi}{4}$ or $\phi = \frac{3\pi}{4}$.

To find the volume we compute $V = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{2\sqrt{2}}{3}\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\phi \, d\phi \, d\theta =$

$\frac{4}{3}\int_0^{2\pi} d\theta = \frac{8}{3}\pi$