MATH 20550 - Calculus III Notes 4

September 29, 2016

This notes include:

- Formulas for Implicit Differentiation
- Directional Derivative
- The Gradient
- Level Surfaces
- Tangent Planes to a Level Surface
- Normal Lines to a Level Surface

14.5 Formulas for Implicit Differentiation

1. Given $F(x, y) = 0$ defines y **implicitly** as a differentiable function of x (loosely speaking, that means y is a function of x but we don't have an explicit formula for y, but rather a complex expression involving x and y,) then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

2. If $z = z(x, y)$ is defined **implicitly** as a function of x, y by the equation $F(x, y, z) = 0$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

14.6 Directional Derivatives and the Gradient Vector

1. If $f = f(x, y)$, then the **gradient** of f is the *vector function* ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

If $f = f(x, y, z)$, then the **gradient** of f is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

2. The **directional derivative** of $f = f(x, y)$ at the point (x_0, y_0) in the direction of a *unit vector* $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

Note that the directional derivative of f is computed in the same manner when $f = f(x, y, z)$.

Remark: f changes (increasing) the fastest in the direction of the gradient vector ∇f. And the maximum rate of change of f is $|\nabla f|$, which is the magnitude of the gradient vector. And f decreases most rapidly in the direction of $-\nabla f$.

3. Recall that a surface S in $\mathbb{R}^3$ can be described using the equation

$$F(x, y, z) = k, \qquad \text{where } k \text{ is a constant.}$$

In this case, S is a **level surface** of a function F of three variables.

Example: The equation $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$ describes a surface in $\mathbb{R}^3$, namely an ellipsoid. Here, $F(x, y, z) = \dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9}$ and $k = 3$.

Aside: The equation $z = f(x, y)$ also gives a surface in $\mathbb{R}^3$. It can be considered as a special case of level surfaces. If we rewrite this equation into $f(x, y) - z = 0$, then we get a level surface of the function $F(x, y, z) = f(x, y) - z$ with $k = 0$. We shall see that there are times we want to do such thing.

Example: $z = 4x^2+y^2$ gives a paraboloid in the xyz-plane. This is exactly a level surface given by $F(x, y, z) = 4x^2 + y^2 - z = 0$.

4. The **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ is the plane that passes through P and has *normal vector* $\nabla F(x_0, y_0, z_0)$.

Recall: The equation of a plane through a point (x_0, y_0, z_0) with <u>normal vector</u> $\mathbf{n} = \langle a, b, c\rangle$ is given by

$$\langle a, b, c\rangle \cdot \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0$$

or

$$\langle a, b, c\rangle \cdot \langle x, y, z\rangle = \langle a, b, c\rangle \cdot \langle x_0, y_0, z_0\rangle$$

5. The **normal line to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ is the line that passes through P and perpendicular to the tangent plane. So, a *parallel vector* of the normal line is given by $\nabla F(x_0, y_0, z_0)$.

Recall: The equation of a line through a point (x_0, y_0, z_0) with parallel vector $\mathbf{v} = \langle a, b, c\rangle$ is given by

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$