MATH 20550 - Calculus III Notes 9

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This notes include:

- Surface Integrals
- Flux Integrals
- Stokes' Theorem
- The Divergence Theorem

16.7 Surface Integrals

1. **Surface Integral of f over the surface S**

   To **compute** the surface integral of f over the surface S, we need a parametrization of the surface S:

   $$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \quad (u,v) \in D.$$

   Then, we apply the formula:

   $$\boxed{\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA.}$$

2. **Surface Integrals of Vector Fields**

   Definition: If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

   $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$$

   This integral is also called the **flux of F across S**.

   To **compute** the above surface integral, again we need a parametrization of the surface S: $\mathbf{r}(u,v)$, where (u,v) lies in the domain D. Then use the formula:

   $$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA.}$$

   Remark on the Orientation of the Surface: Suppose S is an **oriented surface** - loosely speaking, it is a surface that has two distinct sides (you will only be given oriented surfaces in this class). The given choice on the direction of the unit normal vector $\mathbf{n}$ provides S with an **orientation**.

Usually, the problem will indicate that your surface is oriented with the unit normal vector, $\mathbf{n}$, pointing in a certain direction. If your parametrization of the surface, $\mathbf{r}(u, v)$, yields $\mathbf{r}_u \times \mathbf{r}_v$ pointing in the same direction as stated in the problem. Then, you can proceed to compute the flux integral without fear. But, if your parametrization of the surface gives $\mathbf{r}_u \times \mathbf{r}_v$ pointing in the opposite direction of the orientation of the surface, then your answer for the flux integral should be the negative of what you get after going through the computation.

Recall: $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

16.8 Stokes' Theorem

1. **Stokes' Theorem:** Let S be an oriented piecewise-smooth surface with a simple closed boundary curve ∂S that has positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region $\mathbb{R}^3$ that contains S. Then

$$\boxed{\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.}$$

   (i.e. the *line integral of* $\boldsymbol{F}$ *along the boundary curve* ∂S of S is equal to the *flux integral of* curl$\boldsymbol{F}$ *over* S.)

   Recall: The curl of a vector field $\mathbf{F}$ is a vector field. Sometimes, we write curl $\mathbf{F}$ as $\nabla \times \mathbf{F}$.

   The **positive orientation of the boundary curve** ∂S is the direction of ∂S such that if you walk along ∂S in that direction with your head pointing in the same direction as the orientation of S, then the surface S should be on your left.

16.9 The Divergence Theorem

1. **The Divergence Theorem:** Let E be a simple solid region and ∂E is the boundary surface of E, ∂E has outward orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains E. Then

$$\boxed{\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, dV.}$$

   (i.e. the *flux of* $\boldsymbol{F}$ *across the boundary surface* ∂E of E is equal to the *triple integral of* div$\boldsymbol{F}$ *over* E.)

   Recall: div $\mathbf{F}$ is a scalar function.