Contents

1. Tensor and torsion products for PID's

Let R be a commutative ring. Given two R modules A and B, define

$$A \otimes_R B = A \otimes B / \{ra \otimes b = a \otimes rb\}$$

Show that $A \otimes_R B$ is universal for R-bilinear maps $A \times B \to L$, L an R module. Typically for maps and elementary tensors we do not write $\otimes_R$ since it should be obvious what ring we are taking the tensor product over.

Often we will need to restrict to the case in which R is a PID. The key result we have used about $\mathbb{Z}$ modules is that subgroups of free abelian groups are free abelian. For PID's submodules of free modules are free.

A big result for us was that torsion-free abelian groups are flat $\mathbb{Z}$ modules. Look over the proof in Handout 16 and see that the key result was that every submodule of $\mathbb{Z}$ is either 0 or $\mathbb{Z}$ and in the $\mathbb{Z}$ case the inclusion is just $\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}$. The same remark holds for any PID and hence R modules which have no R torsion are R-flat. The definition of R-flat is that A is R-flat if and only if for every injection $0 \to M \to N$, $0 \to A \otimes_R M \to B \otimes_R N$ is exact. It follows that $0 \to M \otimes_R A \to N \otimes_R A$ is also exact. Free R modules have no R torsion.

Just to be clear, $\mathbb{Z}$ torsion means torsion in the usual sense; R torsion means there exists some $r \in R$, $r \neq 0$ such that multiplication by r kills the R-torsion element. So for example fields of finite characteristic have $\mathbb{Z}$-torsion but modules over them have no R torsion.

2. Tensor product of chain complexes

Given two chain complexes of R modules, A_* and B_*, we would like to define their tensor product. You might try defining the n^{th} group as $A_n \otimes_R B_n$ but then you quickly realize that the boundary would lower dimension by 2. A better try is

$$(A_* \otimes_R B_*)_n = \bigoplus_{i+j=n} A_i \otimes_R B_j$$

Then you can do a boundary map using something like $\partial_i^A \otimes 1_{B_j} + 1_{A_i} \otimes \partial_j^B$. Unfortunately, with this precise definition boundary boundary is not zero.

We need to introduce a sign which will annoy us hereafter.

$$\partial_n^{A\otimes B} = \bigoplus_{i+j=n} \partial_i^A \otimes 1_{B_j} + (-1)^i 1_{A_i} \otimes \partial_j^B$$

Now calculate boundary boundary: it suffices to do so on an elementary tensor $a \otimes b$ where a has dimension i and b has dimension j with $i + j = n$.

$$\partial_{n-1}^{A\otimes B}\big(\partial_n^{A\otimes B}(a\otimes b)\big) = \partial_{n-1}^{A\otimes B}\big(\partial_i^A(a)\otimes b + (-1)^i a\otimes \partial_j^B(b)\big) = \partial_{n-1}^{A\otimes B}\big(\partial_i^A(a)\otimes b\big) + (-1)^i \partial_{n-1}^{A\otimes B}\big(a\otimes \partial_j^B(b)\big)$$

$$\partial_{n-1}^{A\otimes B}\big(\partial_i^A(a)\otimes b\big) = \partial_{i-1}^A\big(\partial_i^A(a)\big)\otimes b\big) + (-1)^{i-1}\partial_i^A(a)\otimes \partial_j^B(b) = (-1)^{i-1}\partial_i^A(a)\otimes \partial_j^B(b)$$
$$\partial_{n-1}^{A\otimes B}\big(a\otimes \partial_j^B(b)\big) = \partial_i^A(a)\otimes \partial_j^B(b) + (-1)^i a\otimes \partial_{j-1}^B\big(\partial_j^B(b)\big) = \partial_i^A(a)\otimes \partial_j^B(b)$$

Suppose $f_*\colon A_*\to B_*$ and $g_*\colon C_*\to D_*$ are chain maps. Then

$$f_*\otimes g_*\colon A_*\otimes_R C_*\to B_*\otimes_R D_*$$

defined by $f_*\otimes g_*(a\otimes c) = f(a)\otimes g(c)$ is a chain map.

Theorem 1. *If f_* is chain homotopic to $\bar f_*$ and g_* is chain homotopic to $\bar g_*$ then $f_*\otimes g_*$ is chain homotopic to $\bar f_*\otimes \bar g_*$.*

Proof. It suffices using compositions to prove $f_*\otimes 1$ is chain homotopic to $\bar f_*\otimes 1$ and $1\otimes g_*$ is chain homotopic to $1\otimes \bar g_*$. We do the second and leave the first to the reader.

Let $K_*\colon C_*\to D_{*+1}$ be the chain homotopy. Define $\bar K_n\colon A_i\otimes_R C_j\to A_i\otimes_R C_{j+1}$ by $\bar K_n(a\otimes c) = (-1)^i a\otimes \bar K_j(c)$.

Compute

$$\bar K_{n-1}\big(\partial_n^{A\otimes B}(a\otimes b)\big) = \bar K_{n-1}\big(\partial_i^A(a)\otimes b + (-1)^i a\otimes \partial_j^B(b)\big) = (-1)^{i-1}\partial_i^A(a)\otimes K_j(b) + (-1)^{i+i} a\otimes K_{j-1}\big(\partial_j^B(b)\big)$$

$$\partial_{n+1}^{A\otimes B}\big(K_n(a\otimes b)\big) = (-1)^i\partial_{n+1}^{A\otimes B}\big(a\otimes (K_j(b))\big) = (-1)^i\partial_i^A(a)\otimes K_j(b) + (-1)^{i+i}a\otimes \partial_{j+1}^B\big(K_j(b)\big)$$

so

$$\bar K_{n-1}\big(\partial_n^{A\otimes B}(a\otimes b)\big) + \partial_{n+1}^{A\otimes B}\big(K_n(a\otimes b)\big) = a\otimes g_j(b) - a\otimes \bar g_j(b)$$

□

Corollary 2. *If $A_*^{[0]}$ is chain homotopy equivalent to $A_*^{[1]}$ and if $B_*^{[0]}$ is chain homotopy equivalent to $B_*^{[1]}$ then $A_*^{[0]}\otimes_R B_*^{[0]}$ is chain homotopy equivalent to $A_*^{[1]}\otimes_R B_*^{[1]}$.*

Our very first annoyance comes when we try to compare $A_*\otimes_R B_*$ and $B_*\otimes_R A_*$. If we define $\tau_*\colon A_*\otimes B_*\to B_*\otimes A_*$ by applying the flip map to the elementary tensors, we do not get a chain map. The correct definition is to define

$$\tau_*\colon A_*\otimes_R B_*\to B_*\otimes_R A_*$$

by $\tau_n(a\otimes b) = (-1)^{ij} b\otimes a$ where a has dimension i and b has dimension j with $i+j=n$. To check that this is a chain map it again suffices to check it on elementary tensors.

$$\begin{aligned}
\partial_n^{B\otimes A}\big(\tau_n(a\otimes b)\big) =& (-1)^{ij}\partial_n^{B\otimes A}(b\otimes a) = (-1)^{ij}\left(\partial_j^B(b)\otimes a + (-1)^j b\otimes \partial_i^A(a)\right)\\
\tau_{n-1}\left(\partial_n^{A\otimes B}(a\otimes b)\right) =& \tau_{n-1}\left(\partial_i^A(a)\otimes b + (-1)^i a\otimes \partial_j^B(b)\right) =\\
& (-1)^{(i-1)j} b\otimes \partial_i^A(a) + (-1)^{i+i(j-1)}\partial_j^B(b)\otimes a
\end{aligned}$$

It is still true that τ_* is an involution.

3. A partial Künneth Formula

The goal of this section is to produce a formula for computing the homology of the tensor product of a two short chain complexes.

Fix two abelian groups, or R modules, and pick resolutions by free R modules,

$$0\to F_1\xrightarrow{\partial_1^F} F_0\to A\to 0$$
$$0\to G_1\xrightarrow{\partial_1^G} G_0\to B\to 0$$

Assume R is a PID so this can always be done. Form the tensor product chain complex but put an ϵ on the boundary map where $\epsilon=\pm 1$.

Call the tensor product complex T^ϵ. The complex T^{-1} is the tensor product complex where F_0 is in some even dimension and T^{+1} is the tensor product complex where F_0 is in some odd dimension.

The dimension of G_0 is irrelevant but the dimensions of F_1 and G_1 are one more than the dimensions of F_0 and G_0.

Check that the next diagram commutes.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & F_1 \otimes_R G_1 & \longrightarrow & F_1 \otimes_R G_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \epsilon(1_{F_1}\otimes\partial_1^G)\oplus(\partial_1^F\otimes 1_{G_1})} & & \downarrow{\scriptstyle \epsilon(1_{F_1}\otimes\partial_1^G)} & & \\
0 & \longrightarrow & F_0 \otimes_R G_1 & \longrightarrow & (F_1 \otimes_R G_0) \oplus (F_0 \otimes_R G_1) & \longrightarrow & F_1 \otimes_R G_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\epsilon(1_{F_1}\otimes\partial_1^G)} & & \downarrow{\scriptstyle (\partial_1^F\otimes 1_{G_0})-\epsilon(1_{F_0}\otimes\partial_1^G)} & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes_R G_0 & \longrightarrow & F_0 \otimes_R G_0 & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The horizontal rows are short exact so thinking of the columns as chain complexes, we have a short exact sequence of chain complexes. For notation say $0 \to \mathfrak{A}_* \to T^\epsilon_* \to \mathfrak{B}_* \to 0$. Then

$$0 = H_2(\mathfrak{A}_*) \to H_2(T^\epsilon_*) \to H_2(\mathfrak{B}_*) \to H_1(\mathfrak{A}_*) \to H_1(T^\epsilon_*) \to H_1(\mathfrak{B}_*) \to H_0(\mathfrak{A}_*) \to H_0(T^\epsilon_*) \to H_0(\mathfrak{B}_*) = 0$$

Since $G_1 \xrightarrow{\partial_1^G} G_0$ is a free resolution of B, so is $G_1 \xrightarrow{\pm\epsilon\partial_1^G} G_0$.

Because F_0 is free, $H_k(\mathfrak{A}_*) = \begin{cases} 0 & k \neq 0 \\ F_0 \otimes_R B & k = 0 \end{cases}$. This needs a bit of an argument since the result we proved earlier has the resolution on the right. But τ_* is a chain isomorphism between the resolution on the right complex and the resolution on the left complex.

Similarly, because F_1 is free, $H_k(\mathfrak{B}_*) = \begin{cases} 0 & k \neq 0 \\ F_1 \otimes_R B & k = 1 \end{cases}$.

Hence $H_2(T^\epsilon_*) = 0$ and

$$0 \to H_1(T^\epsilon_*) \to F_1 \otimes_R B \to F_0 \otimes_R B \to H_0(T^\epsilon_*) \to 0$$

The map $F_1 \otimes_R B \to F_0 \otimes_R B$ is $\partial_1^F \otimes 1_B$.

The sequence $0 \to F_1 \xrightarrow{\partial_1^F} F_0 \to A \to 0$ is exact and so as we saw earlier

$$0 \to A *_R B \to F_1 \otimes_R B \xrightarrow{\partial_1^F \otimes_R 1_B} F_0 \otimes_R B \to A \otimes_R B \to 0$$

is exact. Hence with either ϵ, $H_k(T^\epsilon_*) = 0$, $k \neq 0$ or 1. The natural maps

$$A \otimes_R B \longrightarrow H_0(T^\epsilon_*)$$
$$H_1(T^\epsilon_*) \longrightarrow A *_R B$$

are isomorphisms.

This partial result has several useful consequences.

Corollary 3. *$A \otimes B$ and $B \otimes A$ are naturally isomorphic (which we already knew). Moreover $A * B$ and $B * A$ are naturally isomorphic.*

Proposition 4. *Given a map of abelian groups $f\colon A \to B$ and resolutions $0 \to F_1 \to F_0 \to A \to 0$ and $0 \to G_1 \to G_0 \to B \to 0$ be free R modules, there exists a chain map $\hat{f}_*\colon F_* \to G_*$ such that $\hat{f}_*\colon H_0(F_*) = A \to H_0(G_*) = B$ is f. Any two such chain maps are chain homotopic*

Corollary 5. *Given $f\colon A_1 \to A_2$ and $g\colon B_1 \to B_2$. Then there is a chain map*

$$(f,g)_*\colon T^\epsilon_*(A_1, B_1) \to T^\epsilon_*(A_2, B_2)$$

such that

$$\begin{array}{ccc} A_1 \otimes B_1 & \longrightarrow & H_0\big(T^\epsilon_*(A_1, B_1)\big) \\ \downarrow & & \downarrow \\ A_2 \otimes B_2 & \longrightarrow & H_0\big(T^\epsilon_*(A_2, B_2)\big) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H_1\big(T^\epsilon_*(A_1, B_1)\big) & \longrightarrow & A_1 * B_1 \\ \downarrow & & \downarrow \\ H_1\big(T^\epsilon_*(A_2, B_2)\big) & \longrightarrow & A_2 * B_2 \end{array}$$

commute.

3.1. **Direct sum of chain complexes.** Given two chain complexes A_* and B_*, there is a direct sum chain complex $(A \oplus B)_*$ defined in each dimension by $(A \oplus B)_n = A_n \oplus B_n$ and $\partial_n^{A\oplus B} = \partial_n^A \oplus \partial_n^B \colon (A \oplus B)_n \to (A \oplus B)_{n-1}$.

The boundary-boundary verification is routine. Chain maps respect direct sum as do chain homotopy equivalences. Again the required verifications are routine. The flip map $(A \oplus B)_* \to (B \oplus A)_*$ is a chain map.

The direct sum construction can be extended to an arbitrary set of chain complexes.

For any set of R modules A_α, $\alpha \in \mathcal{A}$, and any set of R modules B_β, $\beta \in \mathcal{B}$ the natural map

$$\bigoplus_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} (B_\beta \otimes_R A_\alpha) \to \big(\bigoplus_{\beta\in\mathcal{B}} B_\beta\big) \otimes_R \big(\bigoplus_{\alpha\in\mathcal{A}} A_\alpha\big)$$

is an isomorphism. To describe the isomorphism precisely recall that to define a map out of a direct sum, it is required to give a map from each summand to the range. The map

$$B_{\beta'} \otimes_R A_{\alpha'} \to \big(\bigoplus_{\beta\in\mathcal{B}} B_\beta\big) \otimes_R \big(\bigoplus_{\alpha\in\mathcal{A}} A_\alpha\big)$$

is given by the tensor product of the inclusions $B_{\beta'} \to \oplus_{\beta\in\mathcal{B}} B_\beta$ and $A_{\alpha'} \to \oplus_{\alpha\in\mathcal{A}} A_\alpha$.

There is also a map

$$\big(\bigoplus_{\beta\in\mathcal{B}} B_\beta\big) \otimes_R \big(\bigoplus_{\alpha\in\mathcal{A}} A_\alpha\big) \to \mathop{\times}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} (B_\beta \otimes_R A_\alpha)$$

where $\mathop{\times}\limits_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}}$ denotes the direct product. To describe a map into a direct product, it is necessary to define a map from the domain into each summand of the range. The required maps

$$\big(\bigoplus_{\beta\in\mathcal{B}} B_\beta\big) \otimes_R \big(\bigoplus_{\alpha\in\mathcal{A}} A_\alpha\big) \to B_{\beta'} \otimes_R A_{\alpha'}$$

are given by the tensor product of the projections $\oplus_{\beta\in\mathcal{B}} B_\beta \to B_{\beta'}$ and $\oplus_{\alpha\in\mathcal{A}} A_\alpha \to A_{\alpha'}$. By looking at elementary tensors, we see that the image in the direct product has only finitely many non-zero terms, or in other words it is onto the direct sum. By looking at elementary tensors the map

$$\bigoplus_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} (B_\beta \otimes_R A_\alpha) \to \big(\bigoplus_{\beta\in\mathcal{B}} B_\beta\big) \otimes_R \big(\bigoplus_{\alpha\in\mathcal{A}} A_\alpha\big)$$

is onto and hence split onto and hence is an isomorphism.

These isomorphisms induce isomorphisms of sums of chain complexes. Let A_*^α and B_*^β be sets of chain complexes of R modules. Then

$$\bigoplus_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} (B_*^\beta \otimes_R A_*^\alpha) \to \big(\bigoplus_{\beta\in\mathcal{B}} B_*^\beta\big) \otimes_R \big(\bigoplus_{\alpha\in\mathcal{A}} A_*^\alpha\big)$$

is a chain isomorphism.

4. A decomposition result for chain complexes over a PID

Given a chain complex of R modules, $\{A_*, \partial_*^A\}$ it can be decomposed as follows. For each $k \in \mathbb{Z}$, resolve $H_k(A_*)$ by two free r modules, $0 \to A_{k+1}^{[k]} \xrightarrow{\partial_{k+1}^{A^{[k]}}} A_k^{[k]} \xrightarrow{\rho_k} H_k(A_*) \to 0$. Since $A_k^{[k]}$ is free, there exists a map $A_k^{[k]} \xrightarrow{\iota_k^{[k]}} \ker \partial_k^A \subset A_k$ such that

$$\begin{array}{ccc} A_k^{[k]} & \xrightarrow{\iota_k^{[k]}} & \ker \partial_k^A \subset A_k \\ & \searrow^{\rho_k} & \downarrow \\ & & H_k(A_*) \end{array}$$

commutes. We can then find $\iota_{k+1}^{[k]} \colon A_{k+1}^{[k]} \to A_{k+1}$ such that

$$\begin{array}{ccc} A_{k+1}^{[k]} & \xrightarrow{\iota_{k+1}^{[k]}} & A_{k+1} \\ \Big\downarrow \partial_{k+1}^{A^{[k]}} & & \Big\downarrow \partial_{k+1}^{A} \\ A_k^{[k]} & \xrightarrow{\iota_k^{[k]}} & A_k \end{array}$$

commutes. Make the $A^{[k]}$ into chain complexes by setting the remaining groups to 0. Then $\left\{A_*^{[k]}, \partial_*^{A^{[k]}}\right\}$ is a chain complex, $\iota_*^{[k]}$ is a chain map.

Define $A_\ell^{[\mathfrak{r}]} = \bigoplus_{k\in\mathbb{Z}} A_\ell^{[k]}$ and $\iota_k \colon A_k^{[\mathfrak{r}]} \to A_k$. It then follows that $\iota_* \colon A_*^{[\mathfrak{r}]} \to A_*$ is a quasi-isomorphism. If A_* also consists of free R modules, then ι_* is a chain homotopy equivalence if A_* is bounded below.

Theorem 6. *Let R be a PID. If A_* consists of free R modules and is bounded below, then A_* is chain homotopy equivalent to a chain complex B_* with B_k finitely-generated for all k if and only if $H_k(A_*)$ is finitely-generated for all k.*

5. The full Künneth formula

The Künneth formula computes the homology of the tensor product of two chain complexes. The version we will do is for two chain complexes of free R modules with R a PID.

Theorem 7. *Suppose A_* and B_* are chain complexes of free R modules, R a PID. Suppose A_* and B_* are bounded from below. Then there exists a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(A_*) \otimes H_j(B_*) \to H_n(A_* \otimes B_*) \to \bigoplus_{i+j=n-1} H_i(A_*) * H_j(B_*) \to 0$$

The sequence is split although not naturally.