STIEFEL-WHITNEY HOMOLOGY CLASSES

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1. Introduction

CONSIDER a Z_2-homology manifold M^m without boundary but perhaps not compact. Since the link of every point has the Z_2 homology of the $(m-1)$ sphere, M is an integral Euler space in the sense of Halperin and Toledo [**6**] and thus has homology characteristic classes $s_p(M) \in H_p^{\text{inf}}(M; Z_2)$. To define these classes, consider the sum of all the p-simplices in a barycentric subdivision. This chain is in fact a cycle and the resulting homology class, denoted $s_p(M)$, is a PL invariant [**2**].

On the other hand, a Z_2-homology manifold satisfies Poincaré duality with Z_2 coefficients and so we can define Stiefel-Whitney classes $w_p(M) \in H^p(M; Z_2)$ following Wu [**14**]. To wit, define v_p so that $v_p \cup x = Sq^p x$ for all $x \in H_c^{m-p}(M; Z_2)$ where H_c^* is cohomology with compact supports. Poincaré duality produces a unique $v_p \in H^p(M; Z_2)$ with the above property. $w_p = \sum_{i=0}^{p} Sq^i v_{p-i}$.

Stiefel conjectured that when M is a smooth manifold, $w_p(M)$ and $s_{m-p}(M)$ are Poincaré dual. This was proved by Whitney [**13**], Cheeger [**5**], and Halperin-Toledo [**6**], all by about the same meyhod. In 1973, Blanton and Schweitzer [**3**] almost produced a proof using a new method. Our goal is to use this new method to prove.

THEOREM. *Let M be a Z_2-homology manifold without boundary of dimension m. Then $w_p(M)$ and $s_{m-p}(M)$ are Poincaré dual.*

A Z_2-homology manifold satisfies Poincaré duality with $Z_{(2)}$ coefficients ($Z_{(2)}$ = rationals with odd denominators) where the cohomology groups may have to be taken with twisted coefficients. The $Z_{(2)}$ twisting gives rise to a unique integral twisting, so let $Z^\wedge$ denote the integers with this twist. There is a (non-unique) class in $H_m^{\text{inf}}(M; Z^\wedge)$ which reduces to the fundamental class with twisted $Z_{(2)}$ coefficients. Let [M] denote any such class. Halperin and Toledo have shown that $s_{m-(2p+1)}(M)$ has a natural (untwisted) integral representative, $S_{m-(2p+1)}(M)$. There is a Bockstein $\delta : H^*(M; Z_2) \to H^{*+1}(M; Z^\wedge)$ and we define $W_{2p+1}(M)$ to be $\delta w_{2p}(M)$.

COROLLARY. $W_{2p+1}(M) \cap [M] = S_{m-(2p+1)}(M)$.

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Remarks. Halperin and Toledo define an $S_0(M)$ for all m and prove it is dual to the twisted Euler class for smooth manifolds. Since the Euler class is an unstable class and since as yet we have only a stable tangent bundle for Z_2-homology manifolds, we are unable to deduce a similar result.

M does have a tangent, 2-local, spherical fibration, $M \to BGZ_{(2)}$ (see Quinn [**11**]). $W_{2p+1}(M)$ (resp. $w_{2p}(M)$) is the primary obstruction to factoring this map through $BGZ_{(2)}$ $(k-1)$, $k=2p+1$ (resp. $2p$). $BGZ_{(2)}$ $(k-1)$ is the classifying space for 2-local, $(k-2)$-dimensional, spherical fibrations. This remark will be substantiated in §4.

2. Axioms for homology characteristic classes

Blanton's and Schweitzer's idea was to assume the existance, for manifolds, of homology characteristic classes which satisfied certain axioms. Then they showed that these axioms gave a unique set of such classes for smooth manifolds. The last step is to show that both the s_p and the Poincaré duals of the w_p satisfy the axioms. By altering their axioms slightly we can show uniqueness for Z_2-homology manifolds and we can show that both the s_p and the Poincaré duals of the w_p satisfy the axioms.

So let us assume given classes

$\sigma_p(M) \in H_p^{\inf}(M; Z_2)$ for all p and for all Z_2-manifolds M without boundary. Further assume

(A1) Let $U \subset M$ be an open subset with $f: U \to M$ the inclusion. Then f^*(Poincare dual of $\sigma_p(M)) \cap [U] = \sigma_p(U)$.

(A2) $\sigma_p(M \times N) = \sum_{i=0}^{p} \sigma_i(M) \times \sigma_{p-i}(N)$.

(A3) If the dimension of M is m and if M is compact and connected, $\sigma_0(M) \in H_0^{\inf}(M; Z_2) \cong Z_2$ is just the Euler characteristic of M reduced mod 2.

(A4) $\sigma_m(M)$ is the fundamental class.

(A5) $\beta\sigma_p(M) = \sigma_{p-1}(M)$ when $p \equiv m \pmod 2$ and where β is the Bockstein associated to $0 \to Z_2 \to Z_4 \to Z_2 \to 0$.

(A6) $\sigma_1(RP^2) \neq 0$. Instead of (A5) we could assume

(A′5) For all $m \geq 2$, $\sigma_{m-p}(RP^m) \cap [RP^m] =$ the p-th Stiefel-Whitney class of RP^m for all p.

The Poincaré duals of the w_p are known to satisfy (A1)–(A4) and (A′5). The s_p satisfy (A1)–(A6). (A1) is clear upon a little reflection; (A3) and (A4) are clear. (A2) is in [**7**] and (A5) is in [**6**]. (A6) is a calculation safely left to the reader.

Hence the proof of the theorem comes down to proving that there is a unique set of classes satisfying (A1)–(A6); a unique set satisfying (A1)–(A4) and (A′5); and that these two sets are equal.

3. The proof of the theorem

F. Quinn [**11**] has produced a bundle theory suitable for use with Z_2-homology manifolds. These bundles have a classifying space $B(PL)_K$ in Quinn's notation. We wish K to be all primes except 2 and we denote the resulting space by BH.

Our first step is to produce cohomology characteristic classes $\gamma_p(\xi)$ for $\xi: X \to BH$ a bundle over X, a finite CW complex. Then we produce universal classes $\gamma_p \in H^p(BH; Z_2)$ satisfying

(1) $\xi^* \gamma_p = \gamma_p(\xi)$.

(2) $\mu^* \gamma_p = \sum_{i=0}^{p} \gamma_i \times \gamma_{p-i}$ where $\mu: BH \times BH \to BH$ is the Whitney sum map.

(3) $\gamma_0 = 1$.

(4) $(\tau_M^* \gamma_p) \cap [M] = \sigma_{m-p}(M)$ where $\tau_M: M \to BH$ is the stable tangent bundle.

For (3) to make sense, BH had best be connected. My favorite proof of this is to observe that by transversality, the Hurewicz theorem, and the Thom isomorphism, we have that $H_0(BH; Z_2)$ is isomorphic to the 0 dimensional cobordism group of Z_2-manifolds, which is clearly Z_2. Quinn [**11**] yields a Thom complex, MH, and the necessary transversality theorem.

Since BH is connected, inverse bundles exist over finite complexes (see [**9**] for an adaptable proof). To define τ_M we define instead the normal bundle. In Quinn's notation, $B = M \times I$ and E is a regular neighborhood of B in some large Euclidean space. The required stratification exists ([**12**] Theorem 2.1) and is essentially unique ([**12**] Theorem 3.1).

Notice (4) shows that distinct σ_p's give rise to distinct γ_p's, so that if we can show γ_p is unique we are done.

We relegate the task of producing these $\gamma_p(\xi)$ and γ_p to the last section. There we will also describe an H-map $BO \to BH$ which takes the usual tangent bundle for a smooth manifold to τ_M as defined above. We further promise a map of ring spectra $MO \to MH$ such that the obvious square involving the two Thom isomorphisms commutes.

Since we have a Thom isomorphism we can define Stiefel-Whitney classes $w_p \in H^p(BH; Z_2)$. w_p is $Sq^p(1)$ pulled back via the Thom isomorphism to $H^p(BH; Z_2)$, where 1 is the generator of $H^0(MH; Z_2)$. We prove $\gamma_p = w_p$ by induction on p, which proves the theorem.

First note $\gamma_0 = w_0$ by (3) so we may start the induction. Let $d = \gamma_p - w_p$. Since BH is connected, (2) and the induction hypothesis show

$$\mu^* d = 1 \times d + d \times 1.$$

Hence, in order to show that $d = 0$, we need only show that d vanishes on the indecomposables in $H_*(BH; Z_2)$. There is a ring map $\pi_*(MH) \to H_*(BH; Z_2)$ given by sending M to $(\tau_M)_*[M]$. There is our map $H_*(BO; Z_2) \to H_*(BH; Z_2)$ and we claim that

$$\pi_*(MH) \otimes H_*(BO; Z_2) \to H_*(BH; Z_2) \text{ is onto.}$$

Thus we need only show that d vanishes on the images of $H_*(BO; Z_2)$ and $\pi_*(MH)$.

Granting the claim for now we show that γ_p and w_p agree when pulled back to $H^p(BO; Z_2)$. It follows from Milnor-Stasheff [**10**] that it is enough to show that γ_* (canonical bundle over RP^m) is as expected for all $m \geqslant 2$.

Let ξ be the canonical bundle over RP^m considered as one of Quinn's bundle via the map $BO \to BH$. Then $\tau_{RP^m} = \xi \oplus \cdots \oplus \xi$ $(m+1)$-times. Using (4), (A6) and (A3), one can easily show $\gamma_*(\xi) = 1 + \alpha$ for $m = 2$; α the generator of $H^1(RP^2; Z_2)$. We finish by induction.

We can assume $\gamma_*(\xi) = 1 + \alpha + c\alpha^{m+1}$ in $H^*(RP^{m+1}; Z_2)$. If m is odd, $\gamma_{m+1}(RP^{m+1}) = (c+1)\alpha^{m+1}$. (4) and (A3) show $c = 0$.

If m is even and we have (A′5), we can assume $\gamma_*(\xi) = 1 + \alpha + c\alpha^{m+1} + b\alpha^{m+2}$ in $H^*(RP^{m+2}; Z_2)$. $\gamma_{m+1}(RP^{m+2}) = (\binom{m+3}{m+1} + c)\alpha^{m+1}$ so again $c = 0$. If we have (A5) it is not hard to show $Sq^1\gamma_{2p}(\xi) = \gamma_1(\xi) \cup \gamma_{2p}(\xi) + \gamma_{2p+1}(\xi)$. Since $\gamma_m(\xi) = 0$, $\gamma_{m+1}(\xi) = 0$ and $c = 0$ yet again. This finishes the induction and shows d evaluates zero on $H_p(BO; Z_2)$. By Quinn [**11**], $\pi_p(MH)$ is the cobordism group of unoriented, p-dimensional, Z_2-homology manifolds. If d evaluates non-zero on $\pi_p(MH)$ there exists a p-dimensional, closed, compact Z_2-homology manifold with $\tau^* d \neq 0$. But $\tau_M^* \gamma_p$ is the mod 2 Euler class by (4) and (A3) as is $\tau_M^* w_p$ by [**10**]. Hence $\tau_M^* d = 0$ and the theorem is proved, modulo our claim.

To see the claim, consider $H_*(MH; Z_2)$. This is also $MH_*(HZ_2)$ where HZ_2 is the Eilenberg-MacLane $(Z_2, 0)$ spectrum. We have maps $\pi_*(MH) \to MH_*(HZ_2)$, the usual Hurewicz map, and $e: MH_*(HZ_2) \to A_*$ the Steenrod representation map, where $A_* = H_*(HZ_2; Z_2)$. We shall soon see that e is onto.

Given this the Atiyah-Hirzebruch spectral sequence for $MH_*(HZ_2)$ collapses since it is a multiplicative spectral sequence. A_* is a polynomial algebra [**8**] and hence e is split. A standard argument now shows $\pi_*(MH) \otimes A_* \to H_*(MH; Z_2)$ is an isomorphism.

It is well-known $H_*(MO; Z_2) \to A_*$ is onto. Hence e must be onto and

$\pi_*(MH) \otimes H_*(MO; Z_2) \to H_*(MH; Z_2)$ is onto. $\pi_*(MH) \to H_*(MH; Z_2) \to H_*(BH; Z_2) \to H_*(BH; Z_2)$, where the last map is the one which sends bundles to their inverses, is the map $\pi_*(MH) \to H_*(BH; Z_2)$ in our claim. The map $H_*(MH; Z_2) \to H_*(BH; Z_2)$ above is an isomorphism and our claim easily follows.

4. The proof of the corollary and related remarks

To see that M has a fundamental class in $H_m^{\inf}(M; Z^\wedge)$, let $K \subset M$ be a codimension 0 submanifold with boundary which is compact. There is a Z in $H_m(K, \partial K; Z^\wedge)$ by universal coefficients [**4**]. The map $H_m^{\inf}(M; Z^\wedge) \to H_m(K, \partial K; Z^\wedge)$ maps non-zero to this Z so there is an epimorphism $H_m^{\inf}(M; Z^\wedge) \to Z$ which when tensored with $Z_{(2)}$ is an isomorphism. Split this map and let $[M]$ be the image of 1.

To prove the corollary recall that Halperin and Toledo [**6**] showed $S_{m-(2p+1)}(M) = \beta s_{m-2p}(M)$ where β is now the integral Bockstein. Take the equation $s_{m-2p}(M) = w_{2p}(M) \cap [M]$ from the theorem. We can use the twisted integral class $[M]$ since it reduces right mod. 2. Apply β to both sides of our equation and we get $\beta s_{m-2p} = (\delta w_{2p} M) \cap [M]$ which is the corollary.

Define $W_{2p+1} \in H^{2p+1}(BH; Z_{(2)})$ as δw_{2p}. We can define similar classes in $H^{2p+1}(BGZ_{(2)}; Z^\wedge)$ and $H^{2p+1}(BO; Z)$. It is easy to show $\tau_M^* W_{2p+1} = W_{2p+1}(M)$.

The kernel of $H^k(BGZ)_{(2)}; R) \to H^k(BGZ_{(2)}(k-1); R)$, where R is $Z^\wedge$ or Z_2, is W_{2p+1} or w_{2p} depending on whether k is odd or even. One sees this by using the fact that this is known for BO and $BO(k-1)$ and using the fact that the square

$$\begin{array}{ccc} BO(k-1) & \longrightarrow & BO \\ \downarrow & & \downarrow \\ BG(k-1) & \longrightarrow & BG \end{array}$$

is highly connected. Hence W_{2p+1} or w_{2p} is the primary obstruction to lifting. To be completely precise, the primary obstruction lies in $H^k(BGZ_{(2)}; Z^\wedge_{(2)})$ for k odd, but since W_{2p+1} has order 2 this distinction is rarely important.

If $\tau_M : M \to BGZ_{(2)}$ factors through BG, $W_{2p+1}(M)$ or $w_{2p}(M)$ is the primary obstruction to factoring through $BG(k-1)$. The high connectivity of the above square shows that if τ_M factors through BO we obtain the result of Halperin and Toledo [**6**].

5. The cohomology characteristic classes

In Quinn's description of a bundle over a finite complex X there appears a pair of spaces (E, B) with B a Z_2-homology manifold and a regular neighborhood of X with $\partial B \cap X = \phi$. E is a PL manifold which is a regular neighbourhood of $(B, \partial B)$. We can normalize so that E is parallelizable. This pair has more structure but we have given all we need.

The promised H-map $BO \to BH$ is given by the following natural transformation of vector bundles to Quinn's bundles over finite complexes. Given ξ, a vector bundle over X, a finite complex, let B be $-\xi$ pulled back over a regular neighbourhood of X in some Euclidean space. Let E be ξ pulled back over B. Then (E, B) is a bundle in Quinn's sense with our normalization. This gives a map $BO \to BH$, [**1**]. Since this construction also gives a map $BO(n) \to BH(n)$ weakly compatible with Whitney sum, we also get a map of ring spectra $MO \to MH$ and it is clear that these maps have all the claimed properties.

Given a bundle $\xi : X \to BH$, there is also a bundle $-\xi : X \to BH$. Let (E, B) be a total space for $-\xi$ with our normalization. $\gamma_p(\xi) \in H^p(X; Z_2)$ is the unique class which when pulled into $H^p(B; Z_2)$ is Poincare dual to σ_{b-p} (interior B).

We must show $\gamma_p(\xi)$ is independent of which total space we have chosen for $-\xi$. If $D^r \subset D^{r+k}$ is the standard inclusion of discs, we may replace (E, B) by $(E \times D^{r+k}, B \times D^r)$. By (A2) and (A4), $\gamma_p(\xi)$ is unchanged.

Another change is to have a pair (E, B) over $X \times I$ with total spaces (E_0, B_0) over $X \times 0$ and (E_1, B_1) over $X \times 1$. If we can prove $\gamma_p(\xi)$ is unchanged after replacing (E_0, B_0) by (E_1, B_1) we are done since by a combination of the two types of changes we have considered we can go from any total space for $-\xi$ to any other total space for $-\xi$.

Let (E', B') be $(E_0, B_0) \times [-1, 0] \cup (E, B) \cup (E_1, B_1) \times [1, 2]$. The lemma below proves $\gamma_p(\xi)$ is well-defined.

LEMMA. *Let (E, B) be a total space of a bundle ξ over Y. Let $X \times [0, 1]$ be a subcomplex of Y and suppose that there is a bundle η, with total space (D, A) over X with $(D, A) \times [0, 1] \subset (E, B)$ as codimension 0 submanifolds. Let $f : X \times \frac{1}{2} \to Y$ be the inclusion. Then $\eta = f^*\xi$ and $f^*(\gamma_p(\xi)) = \gamma_p(\eta)$.*

Proof. That $\eta = f^*\xi$ follows from Quinn [**11**]. That $f^*(\gamma_p(\xi)) = \gamma_p(\eta)$ is merely a translation of (A1) applied to $A \times (0, 1) \subset B$.

The lemma is also used to prove naturality. Let $M(f)$ be $X \times [-1, 1] \cup Y \times [0, 1]$ where we have identified $(x, 1)$ and $(f(x), 0)$ for some map $f : X \to Y$. A bundle ξ over Y extends to a bundle over the usual mapping cylinder so that, restricted to Y the bundle is ξ and restricted to $X \times 0$ the bundle is $f^*\xi$. Extend this bundle over $M(f)$ so as to be a product on $Y \times [0, 1]$ and $X \times [-1, 0]$. Then the lemma easily shows

(N) $f^*(\gamma_p(\xi)) = \gamma_p(f^*\xi)$.

Now let $\mathcal{K}$ be the collection of finite subcomplexes of BH. $\mathcal{K}$ is a direct system and over each $K \in \mathcal{K}$ we have the restriction of the universal bundle. Hence we get a homomorphism $\gamma_p(\xi \mid K): H_p(K; Z_2) \to Z_2$. (N) shows that these homomorphisms piece together to get a homomorphism $H_p(BH; Z_2) \to Z_2$ since $H_p(BH; Z_2)$ is the direct limit over $\mathcal{K}$ of the $H_p(K; Z_2)$. Since $H^p(BH; Z_2) \cong \mathrm{Hom}(H_p(BH : Z_2), Z_2)$, we have defined universal classes $\gamma_p \in H^p(BH; Z_2)$. ($N$) now shows 1).

Since the external Whitney sum of (E, B) over X and (E_1, B_1) over Y is just $(E \times E_1, B \times B_1)$ over $X \times Y$, it is easy to see from (A_2) that $\gamma_p(\xi \oplus \eta) = \sum_{i=0}^{p} \gamma_i(\xi) \times \gamma_{p-i}(\eta)$. (2) now follows since any deviation from the claimed formula can be detected by a Whitney sum over a finite complex.

(4) is easily seen from our description of τ_M.

(3) follows from (4) and (A4) since BH is connected.

REFERENCES

1. J. F. Adams, 'A variant of E.H. Brown's respresentability theorem', *Topology* **10** (1971), 185–198.
2. E. Akin, 'Stiefel-Whitney homology classes and bordism', *Trans. Amer. Math. Soc.*, **205** (1975), 341–359.
3. J. D. Blankton and P. A. Schweitzer, 'Axioms for characteristic classes of manifolds', *Proc. Symp. Pure Math.* 27, Differential Geometry, *Amer. Math. Soc.*, 1975, 349–356.
4. G. Bredon, 'Sheaf Theory', McGraw-Hill, New York, 1967.
5. J. Cheeger, '*A combinatorial formula for Stiefel-Whitney classes, in Topology of Manifolds*', J. C. Cantrell and C. H. Edwards (eds), Markham Publ. Co., (1970).
6. S. Halperin and D. Toledo, 'Stiefel-Whitney homology classes', *Ann. of Math.* **96** (1971), 511–525.
7. S. Halperin and D. Toledo, 'The product formula for Stiefel-Whitney homology classes', *Proc. Amer. Math. Soc.* **48** (1975), 239–244.
8. J. W. Milnor, 'The Steenrod algebra and its dual', *Ann. of Math.* **67** (1958), 150–171.
9. J. W. Milnor, 'Microbundles-I'., *Topology* 3, Supplement 1 (1964), 53–80.
10. J. W. Milnor and J. D. Stasheff, 'Characteristic Classes', *Annals of Math. Studies* 76, Princeton Univ. Press, 1974.
11. F. Quinn, '*Semifree group actions and surgery on PL homology manifolds in Geometric Topology*', Lecture Notes in Math. **438** (1975), Springer-Verlag.
12. D. A. Stone, '*Stratified Polyhedra*', Lecture Notes in Math. **252** (1972), Springer Verlag.
13. H. Whitney, 'On the theory of sphere bundles', *Proc. Nat. Acad. Sci.* USA **26** (1940). 148–153.
14. W.-T. Wu, 'Classes caractéristiques et i-carrés d'une variété', *C. R. Acad. Sci. Pans* **230** (1950), 508–511.

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