Massey Triple Products

Laurence R. Taylor
taylor.2@nd.edu

University of Notre Dame
and Institute for Advance Study

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Definition of the Massey triple product

The *Massey triple product* is defined whenever there are three classes $x_i \in H^{m_i}(X; R)$, $i = 1, 2, 3$ such that $x_1 \cup x_2 = 0 = x_2 \cup x_3$. The triple product is a reflection of how these products are zero. To define it, choose cochain representatives for the x_i. Since $\hat{x}_1 \cup \hat{x}_2$ is a coboundary, choose X_{12} so that $\delta(X_{12}) = \hat{x}_1 \cup \hat{x}_2$. Similarly choose X_{23} so that $\delta(X_{23}) = \hat{x}_2 \cup \hat{x}_3$. Form the cochain

$$\mathfrak{m} = X_{12} \cup \hat{x}_3 - (-1)^{m_1} \hat{x}_1 \cup X_{23}$$

and check that it is a cocycle. The choices of X_{12} and of X_{23} are not unique. Precisely either can be altered by a cocycle. Define the ideal

$$\mathfrak{I}^*_{\{x_1\},\{x_3\}} = x_1 \cup H^{*-m_1}(X; R) + H^{*-m_3}(X; R) \cup x_3 \subset H^*(X; R)$$

The cocycle $\mathfrak{m}$ can be altered by any element in $\mathfrak{I}^{m_+ m_2+m_3-1}_{\{x_1\},\{x_3\}}$. For convenience define the annihilator ideal

$$\mathfrak{a}^m_{\{x_1\},\{x_3\}} = \{x \in H^m(X; R) \mid x_1 \cup x = 0 = x \cup x_3\}$$

Definition of the Massey triple product

For any $x_2 \in \mathfrak{a}_{\{x_1\},\{x_3\}}$, define

$$\langle x_1, x_2, x_3 \rangle \in H^{m_1+m_2+m_3-1}(X; R)/\mathfrak{J}^{m_+ m_2+m_3-1}_{\{x_1\},\{x_3\}}$$

to be the class of $\mathfrak{m} = X_{12} \cup \hat{x}_3 - (-1)^{m_1} \hat{x}_1 \cup X_{23}$ in the quotient group.

The Massey triple product can also be thought of as a map

$$\mathfrak{a}^{m_2}_{\{x_1\},\{x_3\}} \longrightarrow H^{m_1+m_2+m_3-1}(X; R)/\mathfrak{J}^{m_+ m_2+m_3-1}_{\{x_1\},\{x_3\}}$$

This map is R-linear. Since $w \cup \langle x_1, x_2, x_3 \rangle \subset \pm\langle x_1, (w \cup x_2), x_3 \rangle$,

the Massey triple product is an $H^*(X; R)$ module map up to sign.

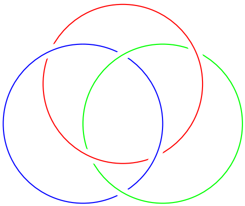
Previous results

Massey seems to have first announced the triple product at the November 1950 meeting of the AMS in Evanston.

Uehara and Massey [15] used the triple product to settle the signs in the Jacobi identity for Whitehead products. (1956)

Massey [7] used the triple product to elucidate some of the cup product structure of sphere bundles. (1958)

The Borromean rings



The Borromean rings Massey [8]. (1968)

More previous results

- Some Massey triple products of Pontryagin classes of normal bundles to foliations vanish. Shulman [12] (1974).
- Massey triple products vanish in a compact Kähler manifold. Deligne, Griffiths, Morgan and Sullivan [2] (1975).
- Massey triple products in an algebraic variety need not vanish mod p. Ekedahl [3] (1983).
- Symplectic manifolds can have non-trivial rational Massey triple products. Babenko & Tamanov [1] and Rudyak & Tralle [11] (2000).
- Non-trivial Massey triple products in Borel cohomology restrict to non-trivial triple products on the fixed sets for symplectic S^1 actions. Stepien & Tralle [13] (2004).
- ...

Difficulties with the Massey triple product

Computing the triple product is a two step process.

- First compute a class $\mathfrak{m} \in \langle x_1, x_2, x_3 \rangle$.
- Then compute $\mathfrak{I}^{m_1+m_2+m_3-1}_{\{x_1\},\{x_3\}}$

  and decide whether $\mathfrak{m} \in \mathfrak{I}_{\{x_1\},\{x_3\}}$ or not.

  Say $\langle x_1, x_2, x_3 \rangle$ is *trivial* if $\mathfrak{m} \in \mathfrak{I}_{\{x_1\},\{x_3\}}$ and *non-trivial* if $\mathfrak{m} \notin \mathfrak{I}_{\{x_1\},\{x_3\}}$.

Naturality has difficulties. Let $f : X \to Y$.
Then

$$f^*(\langle y_1, y_2, y_3 \rangle) \subset \langle f^*(y_1), f^*(y_2), f^*(y_3) \rangle$$

but it can definitely happen that $0 \notin \langle y_1, y_2, y_3 \rangle$ but $0 \in \langle f^*(y_1), f^*(y_2), f^*(y_3) \rangle$. Even worse, it can happen that $0 \notin f^*(\langle y_1, y_2, y_3 \rangle)$ but $0 \in \langle f^*(y_1), f^*(y_2), f^*(y_3) \rangle$.

A way to control the previous difficulties

It is hard to believe that this next idea can really lead to much but I hope to convince you that it does. Suppose the Massey triple product $\langle x_1, x_2, x_3 \rangle$ is defined (which means $x_2 \in \mathfrak{a}_{\{x_1\},\{x_3\}}$).

Suppose further that there is a class x_0 such that $x_0 \cup x_1 = 0 = x_0 \cup x_3$ or equivalently $x_0 \in \mathfrak{a}_{\{x_1\},\{x_3\}}$. Then

$$x_0 \cup \langle x_1, x_2, x_3 \rangle \in H^{m_0+m_1+m_2+m_3-1}(X; R)$$

is a single cohomology class.

Computation: find one element $\mathfrak{m} \in \langle x_1, x_2, x_3 \rangle$ and compute $x_0 \cup \mathfrak{m}$. If $x_0 \cup \mathfrak{m} \neq 0$ then $\langle x_1, x_2, x_3 \rangle$ is non-trivial.

Naturality: Let $f: X \to Y$. Let $\langle y_1, y_2, y_3 \rangle$ be defined and suppose $y_0 \in \mathfrak{a}_{\{y_1\},\{y_3\}}$. Then

$$f^*(y_0 \cup \langle y_1, y_2, y_3 \rangle) = f^*(y_0) \cup \langle f^*(y_1), f^*(y_2), f^*(y_3) \rangle$$

A bonus: new Massey triple products from old ones

Suppose x_0, $x_2 \in \mathfrak{a}_{\{x_1\},\{x_3\}}$. Then x_1, $x_3 \in \mathfrak{a}_{\{x_0\},\{x_2\}}$ and

$$x_0 \cup \langle x_1, x_2, x_3 \rangle = \pm\; x_3 \cup \langle x_0, x_1, x_2 \rangle$$

Remarks: This follows from May [9] or can be proved directly.
The symmetry can be applied twice more: $x_2 \cup \langle x_3, x_0, x_1 \rangle$ and $x_1 \cup \langle x_2, x_3, x_0 \rangle$ are also equal to the first two up to sign and all four are a single cohomology class.
If all the x_i have different degrees, all four Massey triple products are in different dimensions.
By Kraines [5], $\langle x_3, x_2, x_1 \rangle = \pm \langle x_1, x_2, x_3 \rangle$, so one of the equalities is

$$x_0 \cup \langle x_1, x_2, x_3 \rangle = \pm\; x_2 \cup \langle x_1, x_0, x_3 \rangle$$

If $m_0 \neq m_2$, then the two Massey triple products are in different degrees. If $m_0 = m_2$, then both Massey triple products are elements of the same quotient group.

Some results on $x_0 \cup \langle x_1, x_0, x_3 \rangle$

To try to distinguish the two triple products when $m_0 = m_2$, compute $x_0 \cup \langle x_1, x_0, x_3 \rangle$.

Theorem (Milgram [10])

In $H^*(X; \mathbb{Z}/2\mathbb{Z})$

$$x_0 \cup \langle x_1, x_0, x_3 \rangle = x_1 \cup x_3 \cup Sq^{m_2 - 1}(x_0)$$

Theorem (Kraines [5])

If $m_2 = 2m + 1$, then in $H^*(X; \mathbb{Z}/3\mathbb{Z})$,

$$x_0 \cup \langle x_2, x_2, x_2 \rangle = x_0 \cup \beta P^m(x_2)$$

Some results on $x_0 \cup \langle x_1, x_0, x_3 \rangle$

Theorem

If $m_0 = 2m$,

$$2\, x_0 \cup \langle x_1, x_0, x_3 \rangle = 0$$

Proof.

Compute $\langle x_0, x_1, x_0 \rangle$ instead. Since $m_0 = 2m$, $\langle x_0, x_1, x_0 \rangle = -\langle x_0, x_1, x_0 \rangle$, Kraines [5]. Hence $x_3 \cup \langle x_0, x_1, x_0 \rangle = -x_3 \cup \langle x_0, x_1, x_0 \rangle$ $\square$

Corollary

Suppose that $m_0 = m_2 = 2m$. If furthermore $2\, x_0 \cup \langle x_1, x_2, x_3 \rangle \neq 0$, then $\langle x_1, x_0, x_3 \rangle$ and $\langle x_1, x_2, x_3 \rangle$ are distinct and non-trivial.

Massey triple products in manifolds

Let $X = M$ be a closed, compact, connected manifold of dimension n, oriented with coefficients in a field $\mathbb{F}$. *Actually M only needs to be a Poincaré duality space.*
All that is used is that the cup product gives a non-singular pairing

$$\wedge\colon H^{n-r}(M;\mathbb{F}) \times H^{r}(M;\mathbb{F}) \longrightarrow \mathbb{F}$$

Theorem

Cup product gives a non-singular pairing

$$\wedge\colon \mathfrak{a}^{n-r}_{\{x_1\},\{x_3\}} \otimes H^{r}(M;\mathbb{F})/\mathfrak{J}^{r}_{\{x_1\},\{x_3\}} \longrightarrow \mathbb{F}$$

$$\Lambda\colon \mathfrak{a}^{n-r}_{\{x_1\},\{x_3\}} \otimes H^r(M;\mathbb{F})\big/\mathfrak{J}^r_{\{x_1\},\{x_3\}} \longrightarrow \mathbb{F}$$

Proof.

Standard results on pairings reduces the result to showing $\left(\mathfrak{J}^r_{\{x_1\},\{x_3\}}\right)^{\perp} = \mathfrak{a}^{n-r}_{\{x_1\},\{x_3\}}$.
If $x \in \mathfrak{a}^{n-r}_{\{x_1\},\{x_3\}}$ then $x \cup x_1 = 0 = x \cup x_3$. If $y \in \mathfrak{J}^r_{\{x_1\},\{x_3\}}$ then $x \cup y = 0$. Since this holds for all $y \in \mathfrak{J}^r_{\{x_1\},\{x_3\}}$, $x \in \left(\mathfrak{J}^r_{\{x_1\},\{x_3\}}\right)^{\perp}$.
Suppose $z \in \left(\mathfrak{J}^r_{\{x_1\},\{x_3\}}\right)^{\perp}$. Then for all $y \in \mathfrak{J}^r_{\{x_1\},\{x_3\}}$, $z \cup y = 0$. If $z \cup x_1 \neq 0$, there exists $t \in H^*(M;\mathbb{F})$ such that $t \cup z \cup x_1 \neq 0$.
But $t \cup x_1 \in \mathfrak{J}^r_{\{x_1\},\{x_3\}}$ so $t \cup z \cup x_1 = 0$, which is a contradiction.
Hence $z \cup x_1 = 0$ and similarly $z \cup x_3 = 0$. $\square$

A duality between values of Massey triple products

Fix x_1 and x_3 and let $s = m_1 + m_3 - 1$. The Massey triple product defines a linear map

$$\mathfrak{a}^r_{\{x_1\},\{x_3\}} \longrightarrow H^{r+s}(M;\mathbb{F})/\mathfrak{J}^{r+s}_{\{x_1\},\{x_3\}}$$

Let $\mathfrak{M}^r_{\{x_1\},\{x_3\}}$ denote the image in degree r.

Theorem

There is a non-singular pairing

$$\mathfrak{M}^r_{\{x_1\},\{x_3\}} \otimes \mathfrak{M}^{n+s-r}_{\{x_1\},\{x_3\}} \longrightarrow \mathbb{F}$$

Remarks: The pairing is either symmetric or skew-symmetric. If $n+s$ is divisible by 4 (so $r = \frac{n+s}{2}$ is even) the pairing $\mathfrak{M}^r_{\{x_1\},\{x_3\}} \otimes \mathfrak{M}^r_{\{x_1\},\{x_3\}} \to \mathbb{F}$ is skew-symmetric and hence has even rank.

The universal integral example for dimension one

Suppose x_1, x_2 and x_3 are to be 1-dimensional, integral cohomology classes. Then $\langle x_1, x_2, x_3\rangle$ is 2-dimensional. The universal example of this situation is the oriented 5-manifold

$$T^2 \to M^5 \to T^3$$

where the two one dimensional generators of $H^1(T^2;\mathbb{Z})$ transgress to $x_1 \cup x_2$ and $x_2 \cup x_3$ where x_1, x_2 and x_3 are a basis for $H^1(T^3;\mathbb{Z})$. The cohomology is torsion free. The Poincaré series is $1 + 3t + 6t^2 + 6t^3 + 3t^4 + t^5$. By construction $\langle x_1, x_2, x_3\rangle$ is non-trivial. There is a class $\mathfrak{t} \in \mathfrak{a}^3_{\{x_1\},\{x_3\}} \subset H^3(M^5;\mathbb{Z})$ so that $\mathfrak{t} \cup \langle x_1, x_2, x_3\rangle \neq 0$.

- $\mathfrak{M}^2_{\{x_1\},\{x_3\}} \cong \mathbb{Z} \cong \mathfrak{M}^4_{\{x_1\},\{x_3\}}$
- $\mathfrak{M}^4_{\{\mathfrak{t}\},\{x_2\}}$ has rank 2 and the pairing is symmetric.
- There is a basis for $H^2(M^5;\mathbb{Z})$ consisting of elements from distinct triple products.

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