Fibrations, cofibrations and related results, II

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Configuration spaces, braids and applications

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More results in the $M \times \mathbb{R}$ case

To describe the various Thom spaces which go into the decomposition of $\Sigma \mathit{Conf}\,(M \times \mathbb{R}, S)$, begin by discussing 1-dimensional CW complexes. Given a finite set S, an *ordered 1-complex* Γ is a CW complex with vertex set S and a set of edges $\mathcal{E}(\Gamma)$. Each edge is oriented and the set of edges is ordered.

Given an edge $e \in \mathcal{E}(\Gamma)$ define $A_e = A_{s_2,s_1}$ where e starts at vertex s_1 and ends at vertex s_2. Define $A_\Gamma = A_{e_1} \cdots A_{e_k}$ where $e_1, \ldots, e_k$ are the edges of Γ in order. These conventions set up a bijection between products of the A's and ordered 1-complexes. It can be shown that

$$A_\Gamma \neq 0 \text{ if and only if } H_1(\Gamma) = 0.$$

Hence $A_\Gamma \neq 0$ if and only if each path component of Γ is a tree or a single vertex. If we continue the arboreal theme by calling components with single vertices *seeds*, then $A_\Gamma \neq 0$ if and only if Γ is a *forest*.

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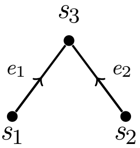

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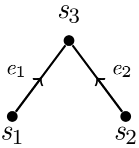
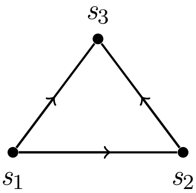


Draw a new edge from s_1 to s_2, provided $e_1 < e_2$, to get the triangle on the next page.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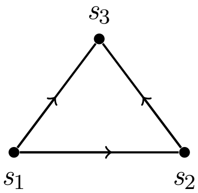
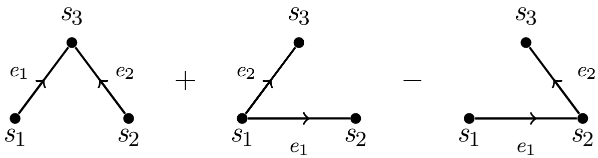


The three-term relation says that a combination of three ordered 1-complexes is 0. They are obtained by combining the three ways of deleting an edge from the triangle, and reordering an edge or two.



$$A_{\Gamma_3} + A_{\Gamma_1} - A_{\Gamma_2} = 0$$

Theorem

Given a vertex which supports a three-term relation then for the three graphs described above

$$H_*(\Gamma_3) \cong H_*(\Gamma_2) \cong H_*(\Gamma_1)$$

A graph partitions its set of vertices by saying two are equivalent if and only if they lie in the same path component. All three graphs yield the same partition.

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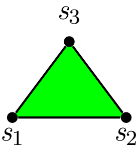

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One basis is given by the *admissible forests*. To define when an forest is admissible, it is first necessary to order S. Then we can orient an edge by starting at the smaller vertex and going to the larger. We can order the edges using lexicographical order. A forest is *admissible* provided no vertex supports an incoming three-term relation using the above orientations and ordering.

Theorem

If $\mathcal{A}(S)$ is the set of admissible forests on the ordered vertex set S then the elements A_Γ for all $\Gamma \in \mathcal{A}(S)$ are an additive basis for $H^*(Conf(\mathbb{R}^n, S); \mathbb{Z})$, $n \geqslant 2$.

For any forest Γ there is a diagonal

$$\Delta_\Gamma \colon X^{\pi_0(\Gamma)} \to X^S$$

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Remark: The admissible basis has an additional property that there is an algorithm for writing any forest as a linear combination of admissible forests.

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Let $S = \{1, 2, 3, 4, 5\}$ and let $\{\{1, 2, 3\}, \{4, 5\}\}$ be a partition. Then a summand of $H^{3(n-1)}(Conf\,(\mathbb{R}^n, S)\,; Z)$ is a tensor product of the top group for 3 points tensor the top group for 2 points.

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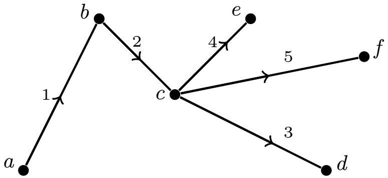
Here is an admissible tree on $\{a, b, c, d, e, f\}$ ordered alphabetically.



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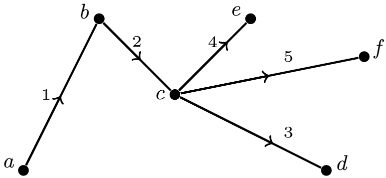
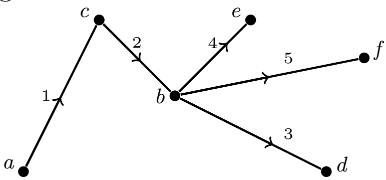


The permutation $\begin{pmatrix} abcdef \\ acbdef \end{pmatrix}$ applied to the above admissible tree gives the tree



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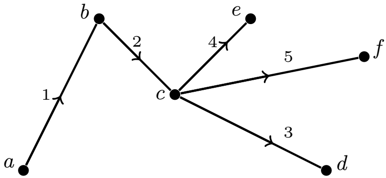

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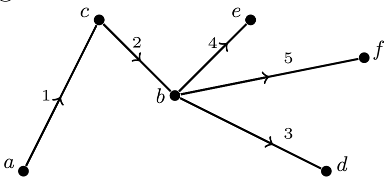

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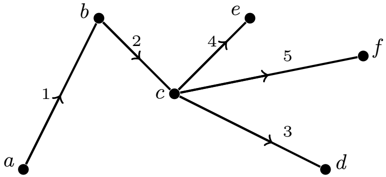

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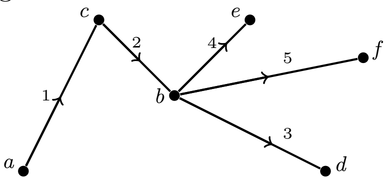

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This shows that the top rep as an integral representation of the of the symmetric subgroup of Σ_S fixing $\mathbf{v}$ is free.

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2. $f_1 \approx f_2$ if there exists a bijection $\phi\colon \mathbf{N} \to \mathbf{N}$ such that $f_1 \circ \phi = f_2$

$C(M, X)$ (canonical identifications)

Since any two countably infinite sets are bijectively equivalent, the choice of set $\mathbf{N}$ is not usually important: if it is we will write $C_{\mathbf{N}}(M, X)$.

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The remark is that two different ϕ's induced the same map on the braid spaces so they may be canonically identified. In the sequel we will write $B_k(M)$ whenever the index set has cardinality k.

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Filter $C(M, X)$ by letting $F_k(M, X) \subset C(M, X)$ be the image of all functions in $E(M, X)$ whose support has at most k elements.

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Define $D_k(M, X)$ to be the cofibre of the inclusion $F_{k-1}(M, X) \subset F_k(M, X)$. If $(X, *)$ is an NDR pair then so is $(F_k(M, X), F_{k-1}(M, X))$ and we can identify the cofibre.

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Therefore map $Conf(M,S) \times_{\Sigma_S} X^S \to D_k(M,X)$ is onto and if $F\Delta \subset X^S$ is the set of points with at least one coordinate the base point, then

$$\left(Conf(M,S) \times_{\Sigma_S} X^S\right) \Big/ \left(Conf(M,S) \times_{\Sigma_S} F\Delta\right) \to D_k(M,X)$$

is a homeomorphism.

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We would like to extend the natural map $\mathfrak{f}_k \colon F_k(M, X) \to D_k(M, X)$ to a map $C(M, X) \to D_k(M, X)$ but this is not usually possible.

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It is however possible to do so stably.

Stable splitting of $C(M, X)$

To describe the extension, first try the most naive thing you (I?) can think of: given f and T any finite set of cardinality k, define $f|_T \colon \mathbf{N} \to M \times X$ by $f|_T(s) = \begin{cases} f(s) & s \in T \\ (m, *) & s \notin T \end{cases}$ where $m \in M$ is any point you like.

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Let $\mathbf{N}' = \binom{\mathbf{N}}{k}$ denote the set of all subsets of $\mathbf{N}$ of cardinality k. Note $\mathbf{N}'$ is also countably infinite.

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If S is in the support of f, let $\langle f \rangle_S \in B_k(M)$ denote the image in $B_k(M)$ of the point in $\mathit{Conf}(M, S)$ given by the composition

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and a commutative diagram

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\begin{tikzcd}
F_k(M,X) \arrow[r, "\mathfrak{f}_k"] \arrow[d, phantom, "\cap"] & D_k(M,X) \arrow[d] \\
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To belabor the point, we could adjoint h_k and note

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The K certainly increases as k increases, but if we pass to the stable world we get an equivalence

$$QC(M,X) \cong Q \bigvee_{k=1}^{\infty} D_k(M,X)$$

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To belabor the point, we could adjoint h_k and note

$$\Sigma^K \mathfrak{f}_k \colon \Sigma^K F_k(M,X) \subset \Sigma^K C(M,X) \xrightarrow{\mathrm{ad} h_k} \Sigma^K D_k(M,X)$$

The K certainly increases as k increases, but if we pass to the stable world we get an equivalence

$$QC(M,X) \cong Q \bigvee_{k=1}^{\infty} D_k(M,X)$$

There are many results concerning this construction and its pieces.

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This generalizes the case in which $M = \mathbb{R}^n$ due to Peter May.

Remark: If X is path connected, then a theorem of Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to a space of sections of a certain bundle $E \to M$.

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Bödigheimer says that $C(M, X)$ is weakly-homotopy equivalent to the space of sections with compact support of $E \to M$.

Corollary

$C(M, X)$ is a proper homotopy invariant of M.