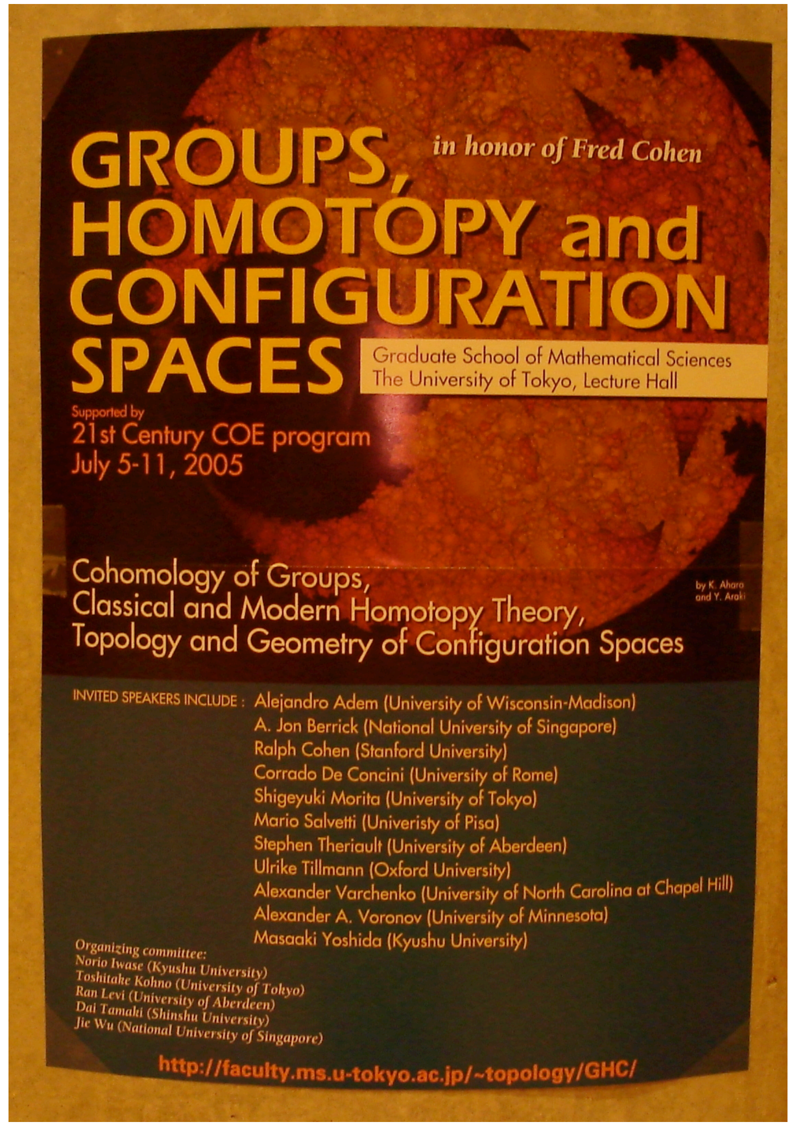
Cohomology of some configuration spaces and associated bundles



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For any space X, $F(X,k) \subset X^k$ denotes k distinct points in X.

Theorem 1.1 (Fadell, Neuwirth 1962)**.** *For a manifold M without boundary*

$$F(M - Q_k) \to F(M, k+\ell) \xrightarrow{\pi} F(M,\ell)$$

is a fibration where π is the restriction of any projection from $k+\ell$ points to ℓ points and Q_k is a subset of k distinct points in M.

Let $M = \mathbb{R}^m$. Then $F(\mathbb{R}^m, 2) = \mathbb{R}^m \times (\mathbb{R}^m - \{0\})$. $A_{21} \in H^{m-1}(F(\mathbb{R}^m, 2); \mathbb{Z})$.

Definition 1.2. Let $\pi\colon \{1,2\} \to \{1,2,\cdots,k\}$ be injective and use π to denote the corresponding projection $\pi\colon M^k \to M^2$. If $i = \pi(2)$ and $j = \pi(1)$, define

$$A_{ij} = \pi^* A_{21}$$

Remarks 1.3. $A_{ij}^2 = 0$ and $A_{ij} = (-1)^{m-1} A_{ji}$.

Theorem 1.4 (Fred). *$H^*\big(F(\mathbb{R}^m, k); \mathbb{Z}\big)$ is the free abelian group generated by the admissible monomials $A_{i_1 j_1} \cdots A_{i_r j_r}$ where a monomial is admissible provided $i_s > j_s$ for $1 \leqslant s \leqslant r$ and $i_1 < \cdots < i_r$.*

When $k = 3$ the generators in degree $2(m-1)$ are $A_{21}A_{31}$ and $A_{21}A_{32}$. What is $A_{31}A_{32}$?

Observation 1.5 (Fred). There exists exactly one Σ_3 invariant relation

$$A_{31}A_{32} = A_{21}A_{32} - A_{21}A_{31} \ .$$

This is the famous *three-term relation.*

Theorem 1.6 (Fred). *As an algebra $H^*\big(F(\mathbb{R}^m, k); \mathbb{Z}\big)$ is the graded commutative algebra on the $\binom{k}{2}$-classes A_{ij}, $1 \leqslant j < i \leqslant k$ of degree $m-1$ subject to the relations*

1. $A_{ij}^2 = 0$
2. $A_{rt}A_{rs} = A_{st}A_{rs} - A_{st}A_{rt}$.

Theorem 1.6 (Fred)**.** *As an algebra $H^*\big(F(\mathbb{R}^m, k);\mathbb{Z}\big)$ is the graded commutative algebra on the $\binom{k}{2}$-classes A_{ij}, $1 \leqslant j < i \leqslant k$ of degree $m-1$ subject to the relations*

1. $A_{ij}^2 = 0$

2. $A_{rt}A_{rs} = A_{st}A_{rs} - A_{st}A_{rt}$.

The equivariant structure is given by

$$\sigma^*(A_{ij}) = \begin{cases} A_{\sigma^{-1}(i)\sigma^{-1}(j)} & \text{if } \sigma^{-1}(i) > \sigma^{-1}(j) \\ (-1)^{m-1}A_{\sigma^{-1}(j)\sigma^{-1}(i)} & \text{if } \sigma^{-1}(i) < \sigma^{-1}(j) \end{cases}$$

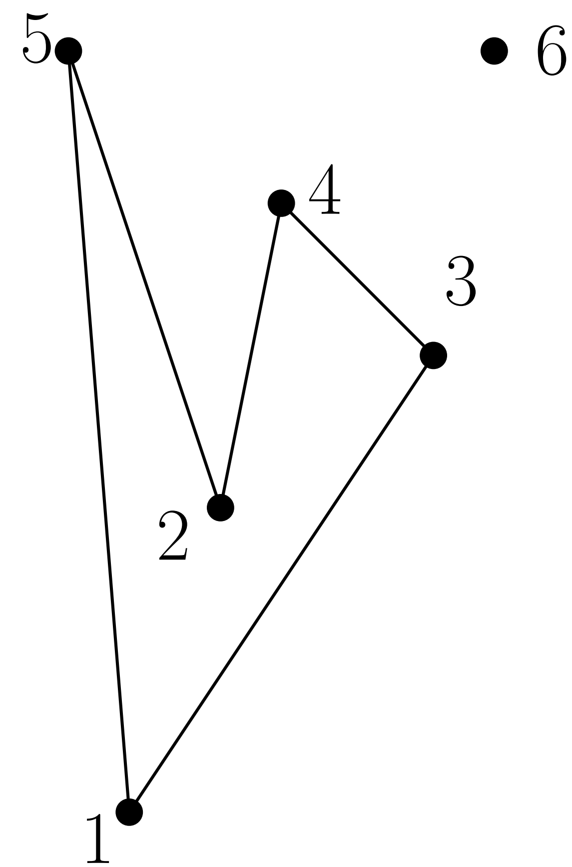
Remark 1.7. As equivariant algebras there are really only two cases, m even and m odd. Let $\mathfrak{A}(k) = H^*\big(F(\mathbb{R}^3, k);\mathbb{Z}\big)$. Tensored with $\mathbb{Z}/2\mathbb{Z}$ there is just one case.

Hereafter most of the discussion will be devoted to the case $\mathfrak{A}(k)$ with only occasional remarks about the case in which m is even.

To every monomial in the A_{ij} associate a graph and vice versa as follows.

Example: Let $k = 6$ and consider $A_{31}A_{42}A_{43}A_{51}A_{52}$.
Put down 6 vertices and add edges.

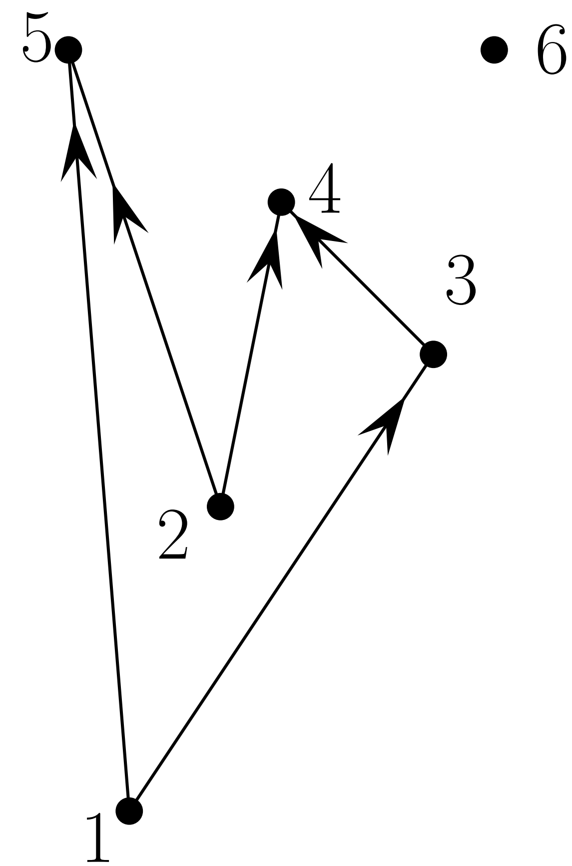
$A_{31}A_{42}A_{43}A_{51}A_{52}$



To every graph Γ on vertex set $\{1, \cdots, k\}$
associate the monomial $A_\Gamma \in \mathfrak{A}(k)$
and vice-versa.

Edges are naturally ordered:

$A_{31}A_{42}A_{43}A_{51}A_{52}$



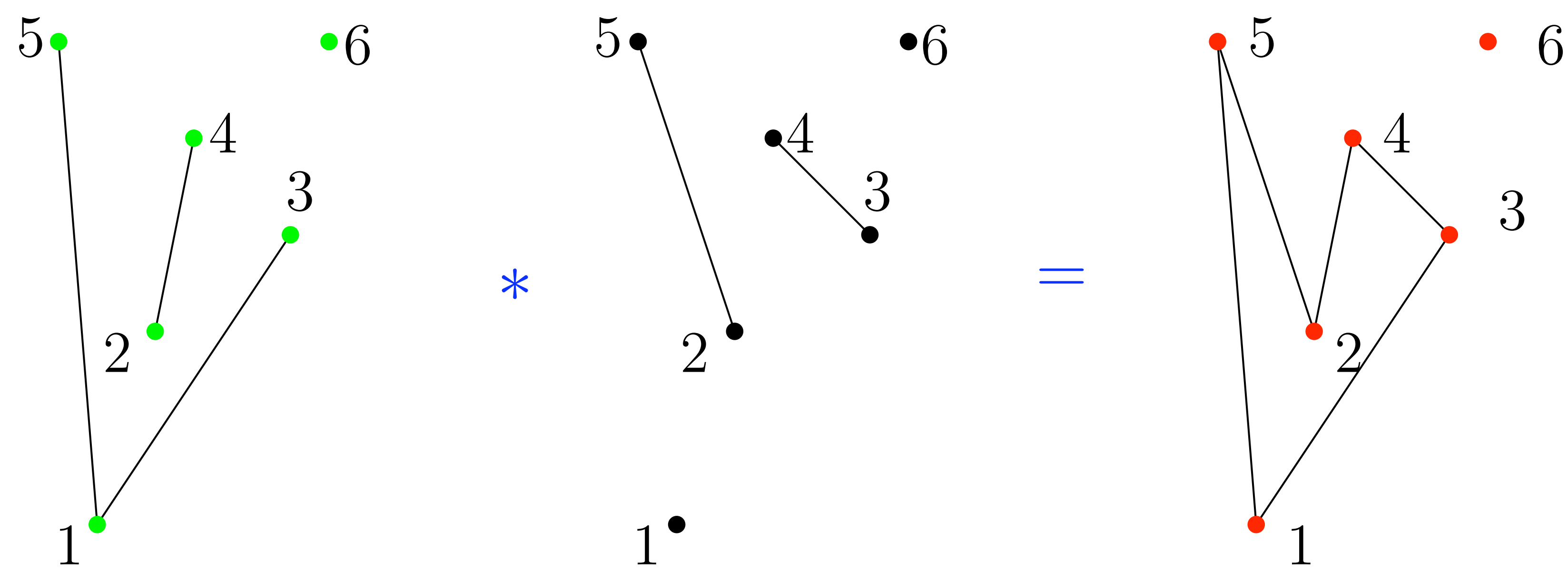
Observation 2.1. A_Γ is admissible if and only if no vertex has more than one *incoming* edge.

Theorem 2.2 (C-T 1993)**.** $A_\Gamma = 0$ *if and only if* $H_1(\Gamma; \mathbb{Z}) \neq 0$.

Remark 2.3. If Γ is admissible then $H_1(\Gamma; \mathbb{Z}) = 0$.

What does multiplication look like graphically?
The simplest guess is correct - just take the collection of the edges.

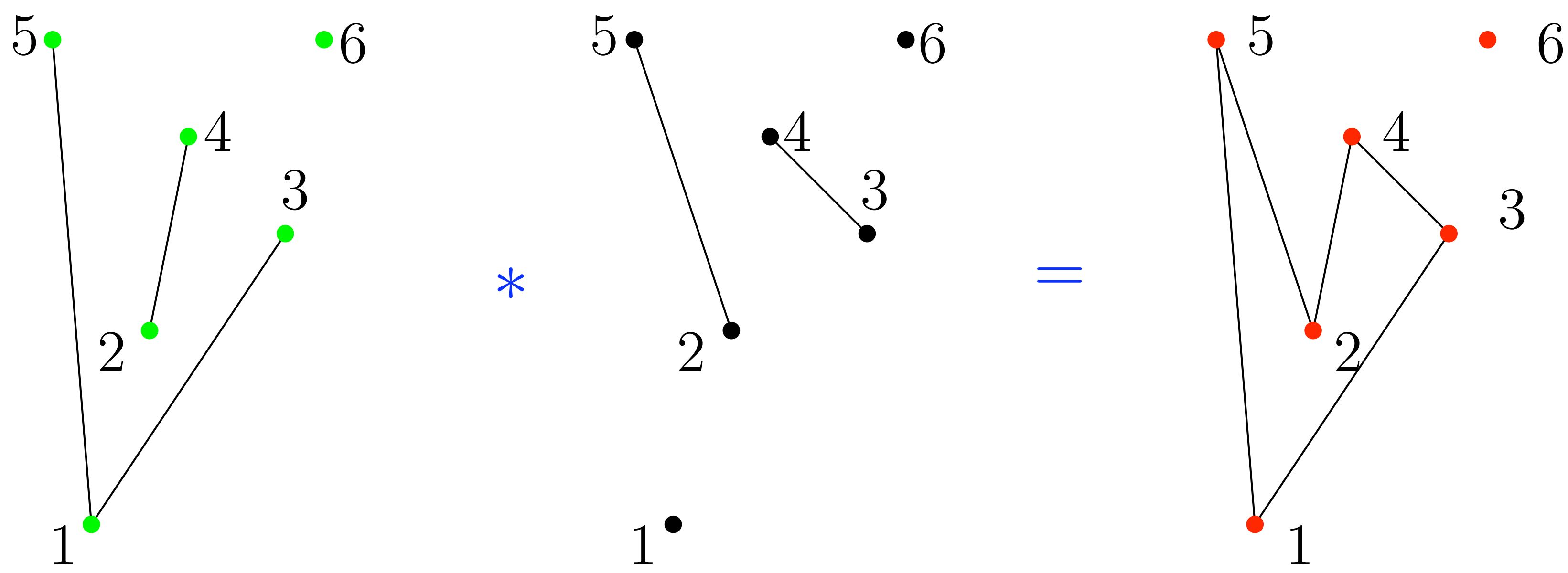
Example:



A product of two admissibles which vanishes.

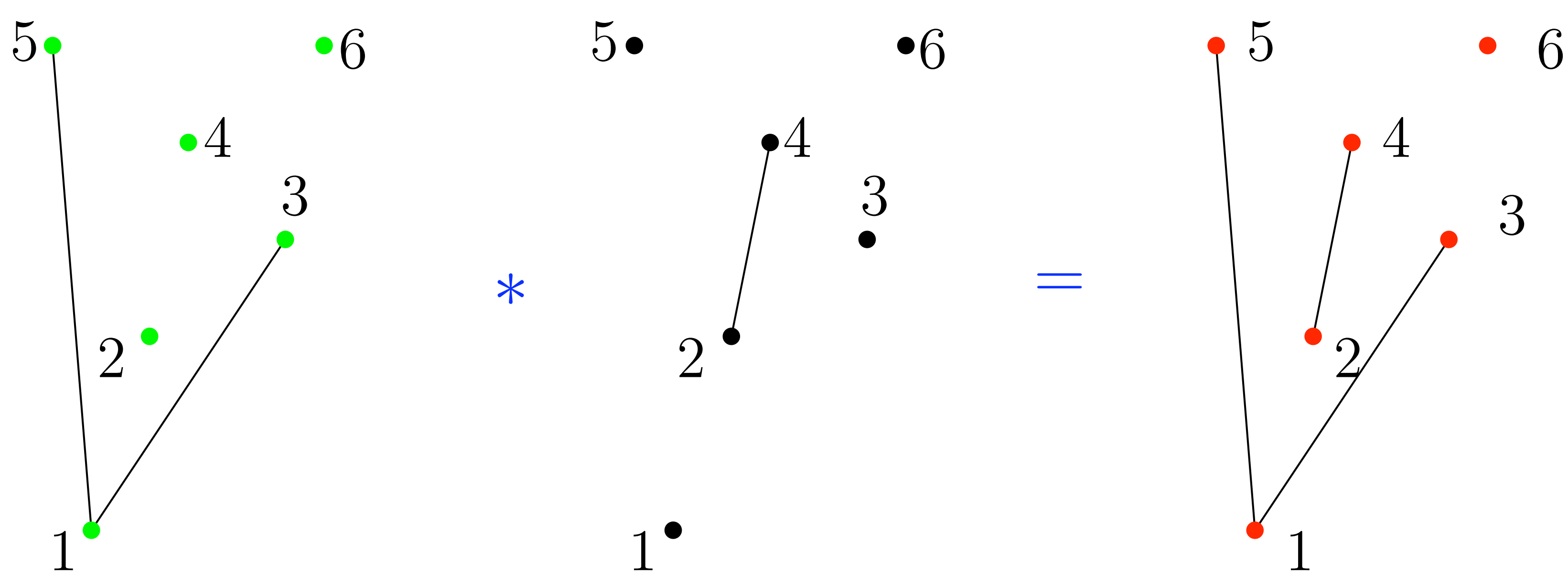
Examples:

1.



A product of two admissibles which vanishes.

2.



Observation 2.4. The monomial associated to any graph is a product of monomials associated to maximal subgraphs each of which is maximal with respect to the property that the union of the edges is connected.

Observation 2.4 suggests studying the graphs which are connected. The top group of $\mathfrak{A}(k)$ occurs in dimension $2(k-1)$ and has a basis consisting of the monomials associated to admissible graphs which are connected. It will be called the *top component representation* and denoted $\mathfrak{A}(k)_{top}$.

Remark 2.5. The top component is the dual to the representation often denote Lie_k.

In [C-T 1992] the entire representation of Σ_k on $H^*\big(F(\mathbb{R}^m, k); \mathbb{Z}\big)$ was built out of the top component representations for $\ell \leqslant k$, tensor products and inductions from certain subgroups called Young subgroups.

One can also prove results using induction from graphs on vertex set $\{1, \cdots, k-1\}$ and some well-chosen maps.

1. $\iota_0 \colon \mathfrak{A}(k-1) \to \mathfrak{A}(k)$ given by adding a disjoint vertex k

2. $\iota_r \colon \mathfrak{A}(k-1) \to \mathfrak{A}(k)$ $1 \leqslant r < k$ given by adding an edge from r to k

Theorem 2.6. *Both maps below are isomorphisms*

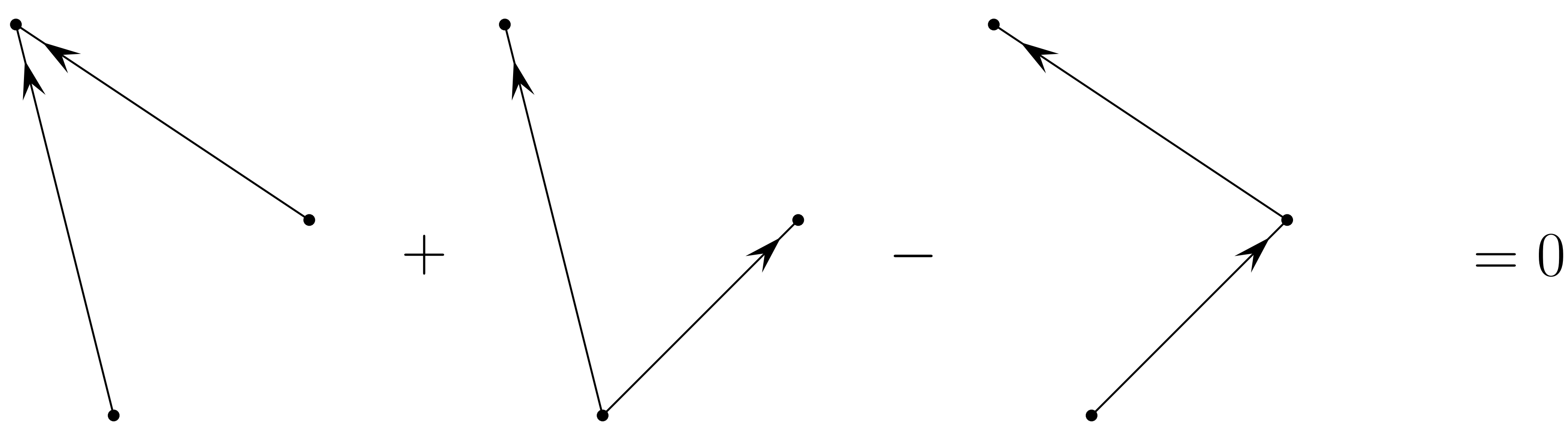
$$\bigoplus_{r=0}^{k-1} \mathfrak{A}(k-1) \xrightarrow{\oplus \iota_r} \mathfrak{A}(k) \qquad\qquad \bigoplus_{r=1}^{k-1} \mathfrak{A}(k-1)_{top} \xrightarrow{\oplus \iota_r} \mathfrak{A}(k)_{top}$$

What does a three-term relation look like graphically?

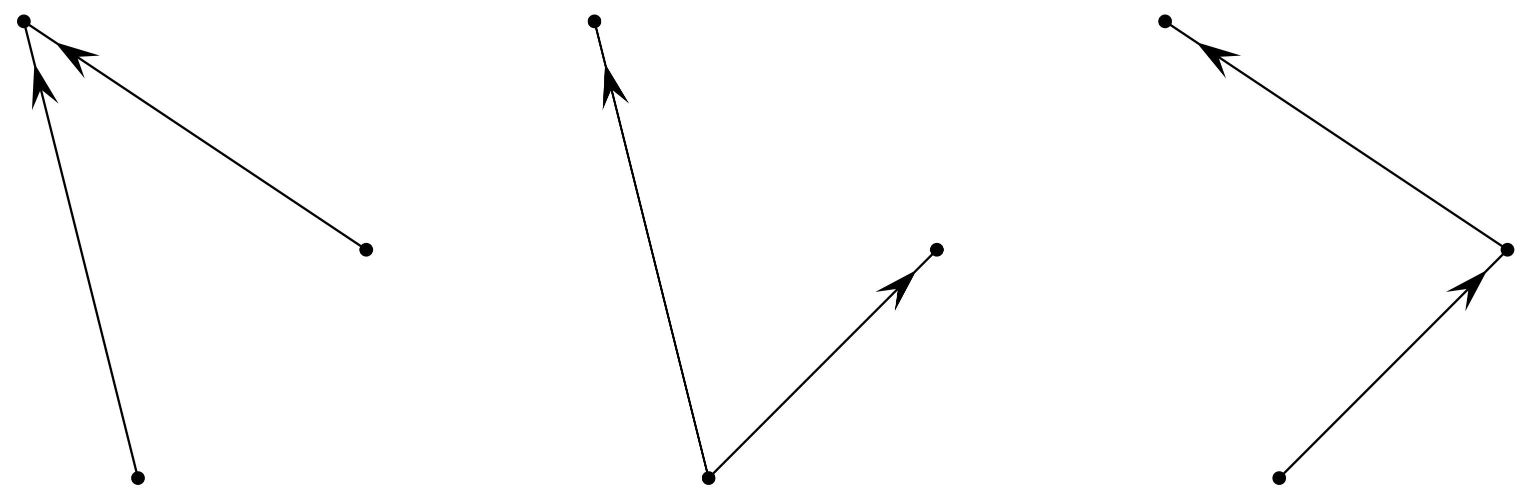
Look at a vertex and 2 edges with distinct initial points coming into it. Add a triangle to get a complex K.

Γ $\qquad\qquad\qquad\qquad$ K

The three-term relation comes from working around the triangle.

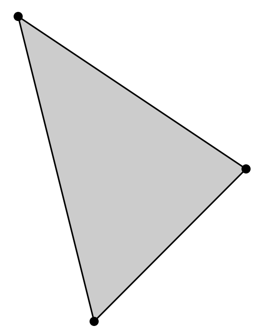


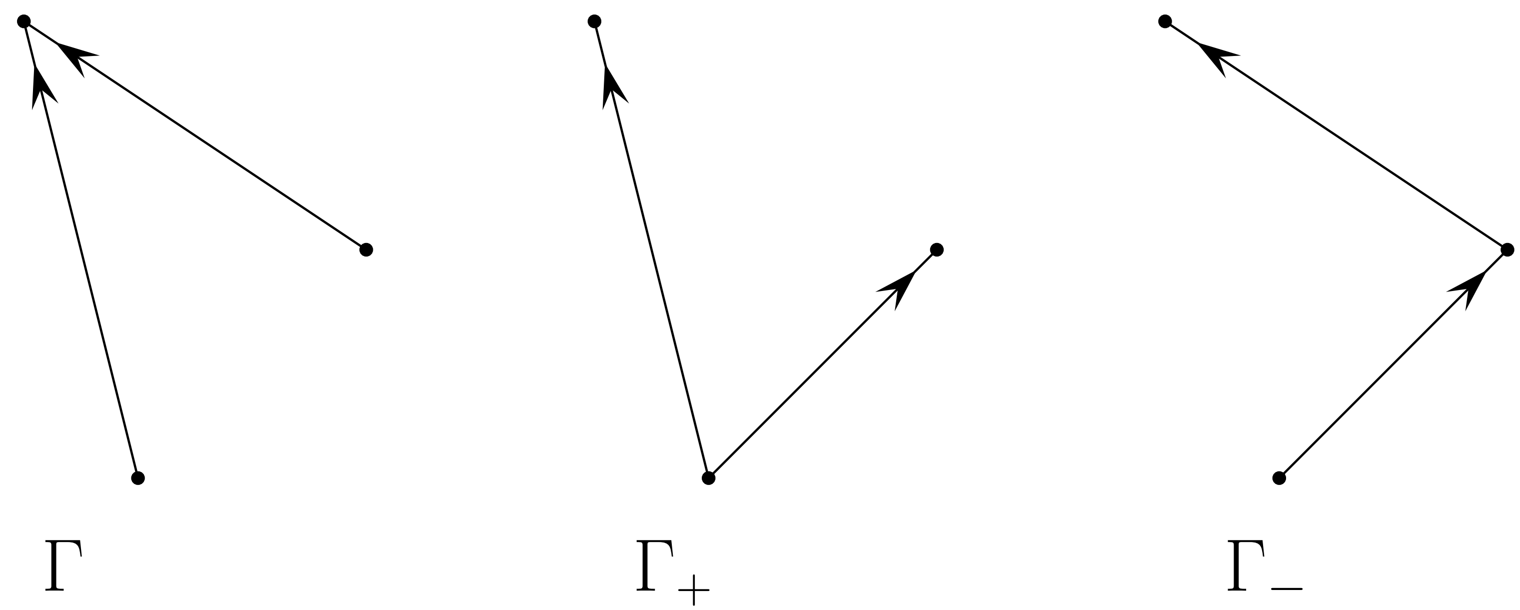
$$A_\Gamma \quad + \quad A_{\Gamma_+} \quad - \quad A_{\Gamma_-} \quad = 0$$



Γ $\qquad\qquad$ Γ_+ $\qquad\qquad$ Γ_-

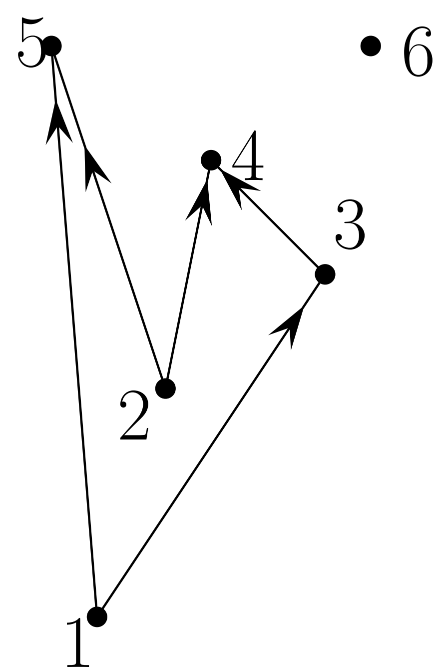
Theorem 2.7 (C-T 1993)**.** $H_*(\Gamma) = H_*(\Gamma_+) = H_*(\Gamma_-)$

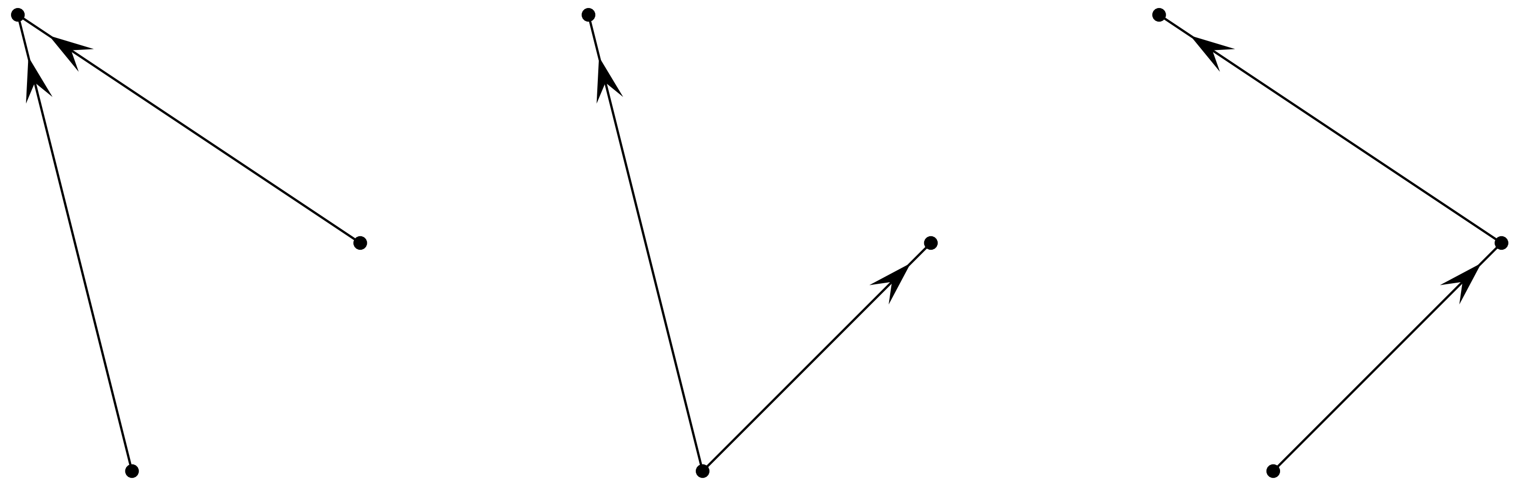
Proof. $= H_*(K)$.  $\square$



Theorem 2.7 (C-T 1993)**.** $H_*(\Gamma) = H_*(\Gamma_+) = H_*(\Gamma_-)$

Define the *incoming weight* of Γ, $\iota_{in}(\Gamma)$, to be the sum of the vertices of the ends of all the edges: define the *outgoing weight* of Γ, $\iota_{out}(\Gamma)$, to be the sum of the vertices of the initial vertices of all the edges.

Example: $\Gamma =$  $\iota_{in}(\Gamma) = 21$ and $\iota_{out}(\Gamma) = 9$.



$\Gamma \qquad\qquad \Gamma_+ \qquad\qquad \Gamma_-$

$$H_*(\Gamma) = H_*(\Gamma_+) = H_*(\Gamma_-)$$
$$\iota_{in}(\Gamma) > \iota_{in}(\Gamma_+) = \iota_{in}(\Gamma_-) > 0$$
$$\iota_{out}(\Gamma) = \iota_{out}(\Gamma_-) > \iota_{out}(\Gamma_+) > 0$$

This shows $A_\Gamma = 0$ if and only if $H_1(\Gamma) \neq 0$.

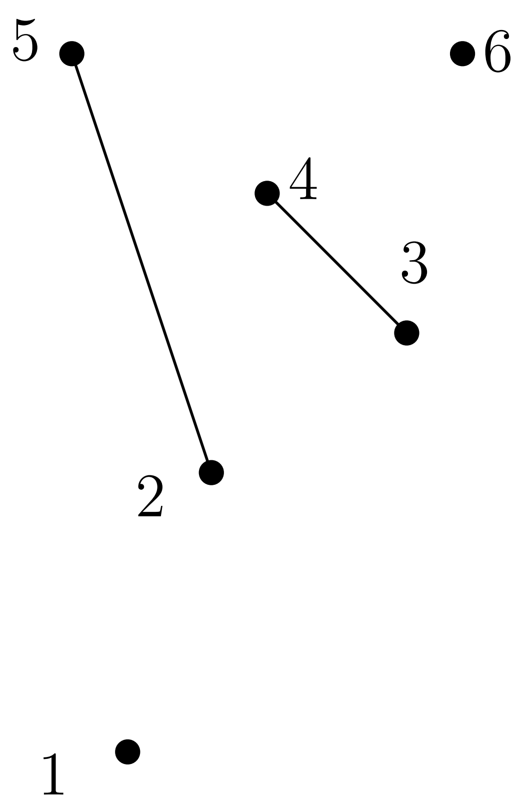
Definition 2.8. For any graph Γ with $H_1(\Gamma) = 0$, define the *admissible expansion* for Γ to be the unique formula in $\mathfrak{A}(k)$

$$A_\Gamma = \sum_\Lambda \alpha_\Lambda(\Gamma)\ A_\Lambda$$

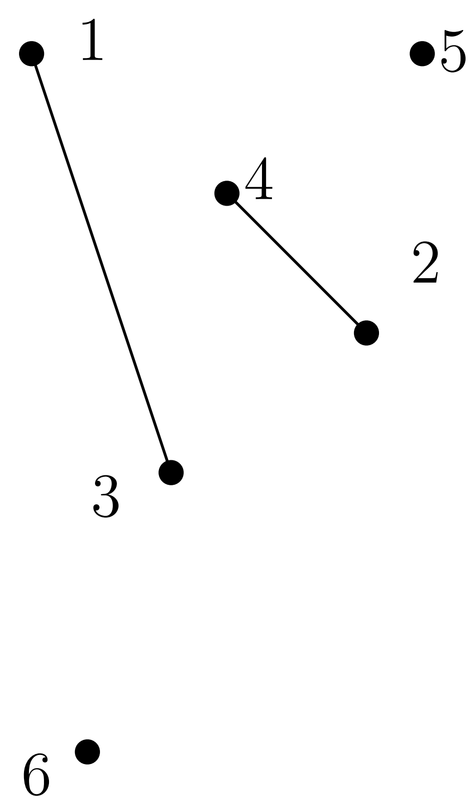
where the sum runs over all admissible graphs.

Given $\sigma \in \Sigma_k$ and a graph Γ define $\sigma^*(\Gamma)$ to be the graph where the vertices are relabeled so that the new vertices are related to the old by $\sigma(v_{new}) = v_{old}$.

Let $\sigma = (23)(156) \in \Sigma_6$.



Γ



$\sigma^*(\Gamma)$

One edge has its orientation reversed so $A_{\sigma^*(\Gamma)} = (-1)\ \sigma^*\big(A_\Gamma\big)$.
In general, if $r(\sigma, \Gamma)$ edges have their orientations reversed,

$$A_{\sigma^*(\Gamma)} = (-1)^{r(\sigma,\Gamma)}\ \sigma^*\big(A_\Gamma\big)\ .$$

In general, $\sigma^*(\Gamma)$ is not admissible even if Γ is, so understanding the Σ_k action is more or less equivalent to understanding the admissible expansion.

Theorem 2.9. *Each coefficient in the admissible expansion of* Γ *is either* 0 *or* ± 1*.*

Remark 2.10. In the admissible basis, the matrix for σ^* has all entries 0 or ± 1.

To prove this result, several formulae involving the expansion of $A_{kj_1} \cdots A_{kj_r}$ in terms of admissible monomials, where $j_1 < \cdots < j_r < k$, are needed.

To describe the first formula, let $P = \{\ell_1, \cdots, \ell_t\} \subset \{2, \cdots, r\}$. Define

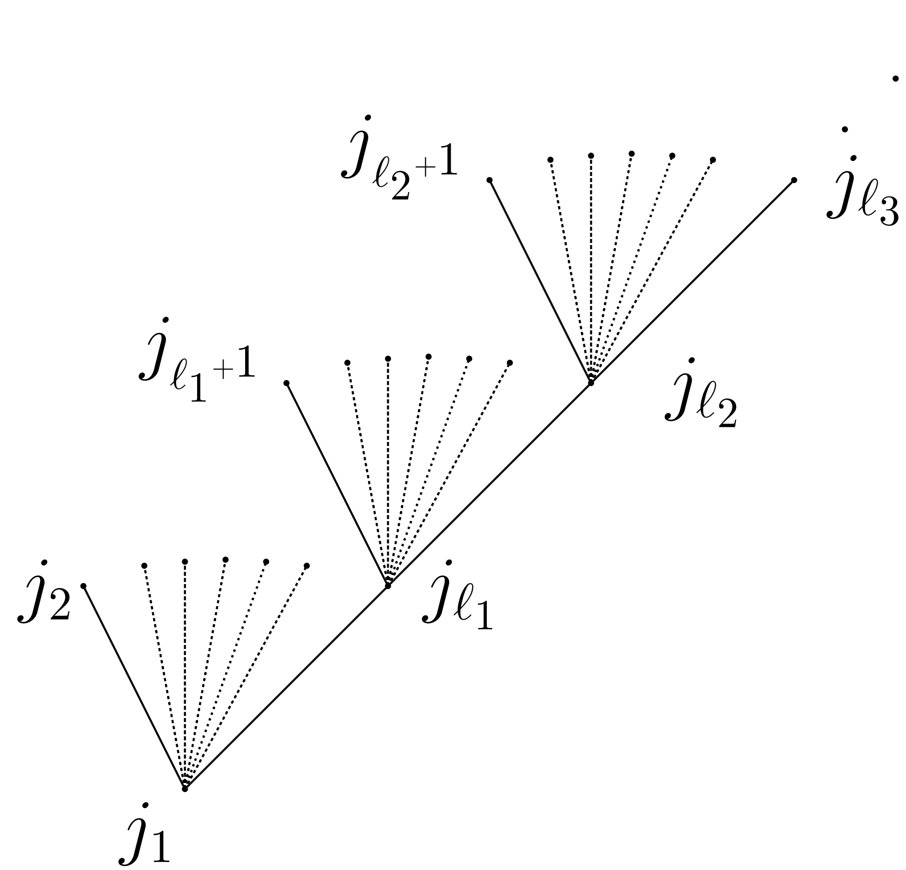
$$A_P = \left(A_{j_2 j_1} A_{j_3 j_1} \cdots A_{j_{\ell_1} j_1}\right) \cdot \left(A_{j_{\ell_1+1} j_{\ell_1}} \cdots A_{j_{\ell_2} j_{\ell_1}}\right) \cdots \left(A_{j_{\ell_t+1} j_{\ell_t}} \cdots A_{j_r j_{\ell_t}}\right)$$

and define Λ_P to be the corresponding graph.

$$P = \{\ell_1, \cdots, \ell_t\} \subset \{2, \cdots, r\}$$

$$A_P = \left(A_{j_2 j_1} A_{j_3 j_1} \cdots A_{j_{\ell_1} j_1}\right) \cdot \left(A_{j_{\ell_1+1} j_{\ell_1}} \cdots A_{j_{\ell_2} j_{\ell_1}}\right) \cdots \left(A_{j_{\ell_t+1} j_{\ell_t}} \cdots A_{j_r j_{\ell_t}}\right)$$

$\Lambda_P =$



The following is immediate.

Lemma 2.11. *For any P as above*

1. Λ_P is admissible and connected;

2. A_P comes from the image of $\mathfrak{A}(k-1)$.

Definition 2.12. For each $r \geqslant 1$ and each ℓ, $1 \leqslant \ell \leqslant r$ define a collection of subsets of $\{2, \cdots, r\}$ inductively as follows: $\mathcal{P}_1(1) = \{\emptyset\}$ and inductively $\mathcal{P}_r(\ell) = \mathcal{P}_{r-1}(\ell)$ for $\ell < r$ and

$$\mathcal{P}_r(r) = \bigcup_{\ell=1}^{r-1} \bigcup_{P \in \mathcal{P}_{r-1}(\ell)} \{P \cup \{r\}\}$$

Example 2.13. For example $\mathcal{P}_r(1) = \{\emptyset\}$ for $r \geqslant 1$; $\mathcal{P}_r(2) = \big\{\{2\}\big\}$ for $r \geqslant 2$; $\mathcal{P}_r(3) = \big\{\{3\}, \{2,3\}\big\}$ for $r \geqslant 3$; $\mathcal{P}_r(4) = \big\{\{4\}, \{2,4\}, \{3,4\}, \{2,3,4\}\big\}$ for $r \geqslant 4$; etc. Note that for $\ell \geqslant 2$, $\mathcal{P}_r(\ell)$ has $2^{\ell-2}$ elements.

Proposition 2.14.

$$(*) \qquad A_{kj_1} \cdots A_{kj_r} = \sum_{\ell=1}^{r} \Big(\sum_{P \in \mathcal{P}_r(\ell)} (-1)^{r-1+|P|} A_P \Big) A_{kj_\ell}$$

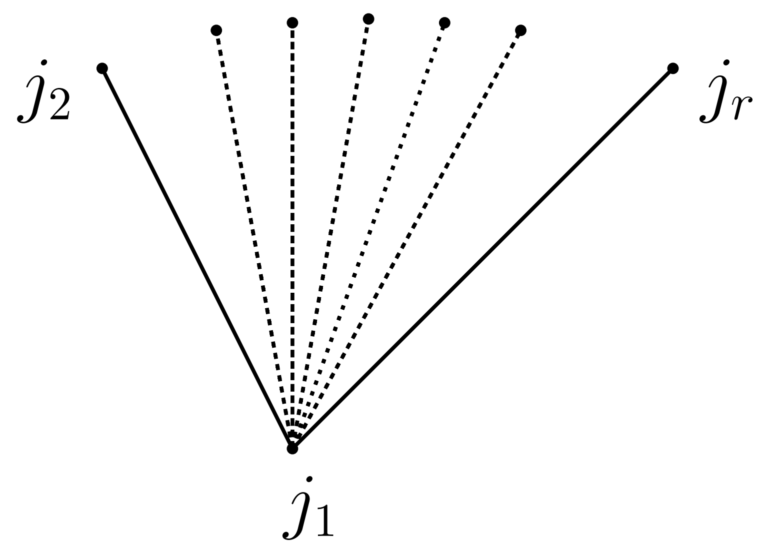
where $\mathcal{P}_r(\ell)$ is defined in 2.12. Each $A_P A_{kj_\ell}$ is admissible so up to distributivity this is the admissible expansion.

Remark 2.15. The coefficient of A_{kj_1} is $\pm A_{\Lambda_\emptyset}$, a single graph.

Here is another way to describe $A_{kj_1} \cdots A_{kj_r}$:

Proposition 2.16. *Let $\tau_\ell = (j_\ell j_1)$ be the indicated transposition. Then*

$$A_{kj_1} \cdots A_{kj_r} = \sum_{\ell=1}^{r} \pm A_{\tau_\ell^*(\Lambda_\emptyset)} \, A_{kj_\ell} \, .$$

$\Lambda_\emptyset \; =$ 

Theorem 2.9 can now be proved inductively.

A further result on the structure of the admissible expansion is

Theorem 2.17. *The sum of the coefficients in the admissible expansion for any graph* Γ *is either* 0 *or* 1 *and it is* 1 *if and only if* Γ *is admissible.*

Given any vector bundle ξ over a space X form the fiberwise configuration spaces: the bundle with base X and fibre $F(\mathbb{R}^m, k)$.

$$F(\mathbb{R}^m, k) \to F(\xi, k) \to X$$

The spectral sequence to calculate the cohomology of the total space behaves similarly to the spectral sequence used to calculate the cohomology of the sphere bundle.

A useful example is to let $X = \mathbb{CP}^\infty$ and to let $\xi = H \oplus \epsilon^1$, where H is the Hopf line bundle and ϵ^1 is a trivial real line bundle.

One sees the that spectral sequence collapses, so let

$$\mathfrak{B}(k) = H^*\big(F(H \oplus \epsilon^1, k); \mathbb{Z}) = \mathfrak{A}(k)[q]$$

be the answer where q is the polynomial class in degree 2 coming from $\mathbb{CP}^\infty$.

The algebra and equivariant structure are not given by the trivial extension to the polynomial algebra.

Here are the relevant formulae for $\mathfrak{B}(k)$.

1. $\sigma^*(q) = q$
2. $A_{ij}^2 = qA_{ij}$
3. $\sigma^*\big(A_{ij}\big) = \begin{cases} A_{\sigma^{-1}(i)\sigma^{-1}(j)} & \text{if } \sigma^{-1}(i) > \sigma^{-1}(j) \\ -A_{\sigma^{-1}(j)\sigma^{-1}(i)} + q & \text{if } \sigma^{-1}(i) < \sigma^{-1}(j) \end{cases}$
4. $A_{rt}A_{rs} = A_{st}A_{rs} - A_{st}A_{rt} + qA_{rt}$ for $t < s < r$.

Observation 2.18. If $\mathfrak{B}(k)$ is filtered by powers of q, the associated graded is $\mathfrak{A}(k) \otimes \mathbb{Z}[q]$ as an equivariant algebra.

Theorem 2.19. *If $\xi \oplus \epsilon^1$ is a vector bundle over X whose dimension is odd then $H^*\big(F(\xi \oplus \epsilon^1, k); \mathbb{Z}\big)$ is given by regrading $\mathfrak{B}(k)$ so the A_{ji} and the q have degree the dimension of ξ; forming $H^*(X; \mathbb{Z}) \otimes \mathfrak{B}(k)$ and adding the relation q equals the Euler class of ξ.*

Remark 2.20. Something similar happens when m is even.

The relation between monomials and graphs still holds and there is still an admissible expansion, but now the coefficients $\alpha_\Lambda(\Gamma)$ are polynomials in q with integer coefficients.

1. For any graph Γ, $A_\Gamma \neq 0$.

Homework: Show that the admissible expansion for the complete graph on k vertices is a power of q times the graph with one edge from 1 to each of $2, \ldots, k$.

The relation between monomials and graphs still holds and there is still an admissible expansion, but now the coefficients $\alpha_\Lambda(\Gamma)$ are polynomials in q with integer coefficients.

1. For any graph Γ, $A_\Gamma \neq 0$.
2. If $H_1(\Gamma) = \mathbb{Z}^{b_1(\Gamma)}$ then $q^{b_1(\Gamma)}$ divides A_Γ and $q^{b_1(\Gamma)+1}$ does not.

Question: What interesting can you say about the highest power of q occurring in the admissible expansion of Γ?

The relation between monomials and graphs still holds and there is still an admissible expansion, but now the coefficients $\alpha_\Lambda(\Gamma)$ are polynomials in q with integer coefficients.

1. For any graph Γ, $A_\Gamma \neq 0$.
2. If $H_1(\Gamma) = \mathbb{Z}^{b_1(\Gamma)}$ then $q^{b_1(\Gamma)}$ divides A_Γ and $q^{b_1(\Gamma)+1}$ does not.
3. The map $\mathfrak{B}(k) \to \mathfrak{A}(k)$ is Σ_k equivariant but is not equivariantly split.