Exotic stratifications

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June 19, 2008

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If (X, A) and (Y, B) are two pairs, then a map $f\colon (X, A) \to (Y, B)$ is said to be *strict*, or *stratum–preserving*, if $f(X \setminus A) \subseteq Y \setminus B$ and $f(A) \subseteq B$. The subspace A of X is said to be *forward tame* if there exists a neighborhood N of A in X and a strict map $H\colon (N \times I,\ A \times I \cup N \times \{0\}) \to (X, A)$ such that $H(x, t) = x$ for all $(x, t) \in A \times I$ and $H(x, 1) = x$ for all $x \in N$.

Let $\mathrm{Map_s}\big((X, A), (Y, B)\big)$ denote the space of strict maps with the compact–open topology. The *homotopy link* of A in X is

$$\mathrm{holink}(X, A) = \mathrm{Map_s}\big(([0, 1], \{0\}),\ (X, A)\big)\ .$$

Evaluation at 0 defines a map $q\colon \mathrm{holink}(X, A) \to A$ which should be thought of as a model for a normal fibration of A in X.

The pair (X, A) is said to be a *homotopically stratified pair* if A is forward tame in X and if $q\colon \mathrm{holink}(X, A) \to A$ is a fibration. If in addition, the fiber of $q\colon \mathrm{holink}(X, A) \to A$ is finitely dominated, then (X, A) is said to be *homotopically stratified with finitely dominated local holinks.* If the strata A and $X \setminus A$ are manifolds (without boundary), X is a locally compact separable metric space, and (X, A) is homotopically stratified with finitely dominated local holinks, then (X, A) is a *manifold stratified pair*.

(1) From a smooth embedding $B \to W$ we construct a vector bundle over B of dimension k, the codimension of the embedding.
(2) Vector bundles over B of dimension k are classified by maps of B into a classifying space.
(3) There is a smooth embedding of B into the total space of any vector bundle.
(4) There is an embedding of the total space of the vector bundle into W which is unique up to isotopy.
(5) All dimension k vector bundles over B occur as a normal bundle to some codimension k embedding.

Define a *controlled map* from $q\colon Y \to B$ to $p\colon X \to B$: $F\colon q \to p$ to be a level-preserving map

$$F\colon Y \times [0,1) \to X \times [0,1)$$

such that the map

$$\hat{F}\colon Y \times [0,1] \to B \times [0,1]$$

is continuous where

$$\hat{F}(y,t) = \big(p \times 1_{[0,1)}\big) \circ F(y,t)$$

if $0 \leqslant t < 1$ and

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An *approximate fibration* is a map $p\colon X \to B$ with the *controlled homotopy lifting property.* A *manifold approximate fibration* or **MAF** is a map $p\colon M \to B$ where M and B are paracompact Hausdorff manifolds without boundary, p is a proper map, and p is an approximate fibration.

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The *fibre germ* of a **MAF** $p\colon M \to B$ is the **MAF** given by restriction

$$\mathfrak{p}\colon p^{-1}(U) \to U$$

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For a fixed fibre germ $\mathfrak{p}\colon V \to \mathbb{R}^i$, there is a classifying space $\mathbf{MAF}(\mathfrak{p})$ and a fibration

$$\mu\colon \mathbf{MAF}(\mathfrak{p}) \to \mathbf{BTOP}(i)$$

The fibre of μ is

$$\mathbf{BTOP}^c(V \to \mathbb{R}^i)$$

The space of **MAF**'s over B with fibre germ $\mathfrak{p}$ is homotopy equivalent to the space of lifts

$$\begin{array}{ccc} & & \mathbf{MAF}(\mathfrak{p}) \\ & \nearrow & \Big\downarrow \mu \\ B & \xrightarrow{\tau_B} & \mathbf{BTOP}(i) \end{array}$$

where τ_B classifies the tangent bundle to B provided $\dim V \geqslant 6$.

Let $p\colon M \to B \times \mathbb{R}$ be a map. The *tear-drop* of p is the set $T(p) = M \amalg B$ with the tear-drop topology. The tear-drop topology is the minimal topology such that $M \subset T(p)$ is an open embedding and the function $c\colon T(p) \to B \times (-\infty, \infty]$ is continuous where $c(x) = p(x)$ for all $x \in M$ and $c(b) = (b, \infty)$ for all $b \in B$.

Theorem 4.1. *The tear-drop $T(p)$ is a manifold stratified space with two strata if and only if p is a* **MAF**.

Theorem 4.2. *If (X, B) is a manifold stratified space with two strata with $\dim X \geqslant 6$, then there is a* **MAF** *$p\colon M \to B \times \mathbb{R}$ and an embedding $T(p) \subset X$ which is the identity on B and whose image contains a neighborhood of B.*

Actually with more work, Hughes proved 4.1 and 4.2 without the two-strata hypothesis.

$$
\begin{array}{ccc}
\mathbf{MAF}(\mathfrak{p}) & \xrightarrow{\iota} & \mathbf{MAF}(\mathfrak{p} \times 1_{\mathbb{R}}) \\
\downarrow & & \downarrow \\
\mathbf{BTOP}(k) & \rightarrow & \mathbf{BTOP}(k+1)
\end{array} \tag{5.3}
$$

$$
\begin{array}{ccccc}
\mathbf{MAF}(\mathfrak{p}) & \rightarrow & E(\mathfrak{p} \times 1_{\mathbb{R}}) & \rightarrow & \mathbf{MAF}(\mathfrak{p} \times 1_{\mathbb{R}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{BTOP}(k) & = & \mathbf{BTOP}(k) & \rightarrow & \mathbf{BTOP}(k+1)
\end{array}
$$

$$
\begin{array}{ccc}
\mathbf{BTOP}^c(\mathfrak{p}) & \rightarrow & \mathbf{BTOP}^c(\mathfrak{p} \times 1_{\mathbb{R}}) \\
\downarrow & & \downarrow \\
\mathbf{MAF}(\mathfrak{p}) & \rightarrow & E(\mathfrak{p} \times 1_{\mathbb{R}}) \\
\downarrow & & \downarrow \\
\mathbf{BTOP}(k) & = & \mathbf{BTOP}(k)
\end{array}
$$

Theorem 5.5. *If $p\colon M \to B \times \mathbb{R}$ is a* **MAF**, *the tear-drop $T(p \times 1_{\mathbb{R}})$ has a mapping cylinder neighborhood.*

Corollary 5.6. *If (X, B) is a two-stratum manifold stratified space with tear-drop neighborhood $T(p)$ then $(X \times \mathbb{R}, B \times \mathbb{R})$ is a two-stratum manifold stratified space with a mapping cylinder neighborhood.*

We say the fibre germ is *trivial* when it is of the form $\mathfrak{p}\colon V \times \mathbb{R}^i \to \mathbb{R}^i$ where V is some compact manifold without boundary.

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When the fibre germ is trivial, Anderson & Hsiang show that the fibre is the space of bounded concordances, $C^b(V \times \mathbb{R}^i \to \mathbb{R}^i)$ is the fibre of the stabilization map

$$\mathbf{MAF}(\mathfrak{p}) \to E(\mathfrak{p} \times 1_{\mathbb{R}})$$

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Theorem 5.7. *If* $i + \dim V \geq 6$, *then there exists a group isomorphism*

$$\alpha_k \colon \pi_k\big(C^b(V \times \mathbb{R}^i \to \mathbb{R}^i)\big) \longrightarrow \begin{cases} \mathrm{Wh}_1(\mathbb{Z}\pi_1 F) & \textit{if } k = i-1 \\ \widetilde{K}_0(\mathbb{Z}\pi_1 F) & \textit{if } k = i-2 \\ K_{2+k-i}(\mathbb{Z}\pi_1 F) & \textit{if } 0 \leqslant k < i-2. \end{cases}$$

Corollary 5.8. *(Edwards) If* $B \subset W$ *is locally-flat and dimension* $B \geqslant 5$ *then the embedding has a mapping cylinder neighborhood.*

$$\begin{array}{ccc}
\mathbf{BTOP}(V) & \longrightarrow & \mathbf{BTOP}(V) \times \mathbf{BTOP}(i) \\
\downarrow \times 1_{\mathbb{R}^i} & & \downarrow \Psi \\
\mathbf{BTOP}^c(V \times \mathbb{R}^i) & \longrightarrow & \mathbf{MAF}(\mathfrak{p})
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\mathbf{BTOP}^c(V \times \mathbb{R}^i) & \longrightarrow & \mathbf{MAF}(\mathfrak{p})
\end{array}
$$

Theorem 6.9. *For each integer $m \geqslant 5$, there exists a closed compact m-manifold V and a* **MAF** *over $p\colon W \to S^1$ with fibre-germ $\mathfrak{p}\colon V \times \mathbb{R} \to \mathbb{R}$ such that the* **MAF** *over S^1 with fibre-germ $V \times \mathbb{R}^i \times \mathbb{R} \to \mathbb{R}^i \times \mathbb{R}$ is not controlled homeomorphic to a fibre bundle for any integer $i \geqslant 0$.*

A **MAF** $p\colon M \to S^1$ with trivial fibre-germ, is determined by an element $h\colon \pi_0\big(\mathbf{TOP}^b(V \times \mathbb{R})\big)$.

There exists a crossed homomorphism

$$\beta\colon \pi_0\big(\mathbf{TOP}^b(V \times \mathbb{R})\big) \to \mathrm{Wh}(\mathbb{Z}\pi_1 V)$$

defined by using the bounded homeomorphism to construct an inertial h-cobordism and then taking the torsion.

Theorem 7.10. *Let* $h \in \pi_0\big(\mathbf{TOP}^b(V \times \mathbb{R})\big)$ *and let* $p\colon M \to S^1$ *be the associated* **MAF** *with* $\dim M \geqslant 6$.

(1) *The following are equivalent.*
 (a) p *is controlled homeomorphic to a fibre bundle projection with fibre* V.
 (b) $\beta(h) = 0 \in \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.

(2) *The following are equivalent.*
 (a) $p \times 1_{\mathbb{R}}$ *is controlled homeomorphic to a fibre bundle projection with fibre* V.
 (b) $\beta(h) \in \mathit{Im}\, N \subset \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.

(3) *There exists a subgroup* G *of* $\widetilde{K}_0(\mathbb{Z}\pi_1 V)$ *and a function*

$$N_0\colon G \to \mathrm{Wh}(\mathbb{Z}\pi_1 V)/\mathit{Im}\, N$$

such that the following are equivalent.
 (a) $p \times 1_{\mathbb{R}^2}$ *is controlled homeomorphic to a fibre bundle projection with fibre* V.
 (b) $\beta(h) \in \mathrm{Wh}(\mathbb{Z}\pi_1 V)/\mathit{Im}\, N$ *is in* $N_0(G)$.

Theorem 8.11. *Let* $h \in \pi_0\big(\mathbf{TOP}^b(V \times \mathbb{R})\big)$ *and let* $p\colon M \to S^1$ *be the associated* **MAF** *with* $\dim M \geqslant 6$.

(1) *The following are equivalent.*

  (a) *p is controlled homeomorphic to a fibre bundle projection.*

  (b) $\beta(h) \in \mathit{Im}\,(1 - h_*) \subset \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.

(2) *The following are equivalent.*

  (a) *$p \times 1_{\mathbb{R}}$ is controlled homeomorphic to a fibre bundle projection.*

  (b) $\beta(h) \in \mathit{Im}\,N + \mathit{Im}\,(1 - h_*) \subset \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.

(3) *If $\widetilde{K}_k(\mathbb{Z}\pi_1 V) = 0$ for $k \leqslant 0$ then the following are equivalent.*

  (a) *$p \times 1_{\mathbb{R}^i}$ is controlled homeomorphic to a fibre bundle projection.*

  (b) $\beta(h) \in \mathit{Im}\,N + \mathit{Im}\,(1 - h_*) \subset \mathrm{Wh}(\mathbb{Z}\pi_1 V)$.