The Story Behind Ponytail Motion

It’s no surprise this article\textsuperscript{1} is the most read on the SIAM Journal on Applied Mathematics website. The author models a commonly observed phenomenon, the swaying of a jogger’s ponytail, and mathematically describes the resulting behavior. The article is clear, relatable, and applies multiple physical and mathematical models to the phenomenon.

The article begins by introducing the physical model, which is common to oscillating mechanical or electrical systems: that of parametric excitation, where the hanging body is unstable to lateral perturbations. First, the ponytail is modeled as a rigid rod: a pendulum rotating about a point. In this model, any small angular deviation $\theta(t)$ from the vertical position satisfies Hill’s\textsuperscript{2} equation, which is a linear second order ordinary differential equation with a periodic coefficient\textsuperscript{3}. This equation has one solution which grows exponentially in time if the natural frequency of the pendulum is close to an integer multiple of half the frequency of oscillation of the support\textsuperscript{4}. Then, the purely vertical motion of the pendulum is unstable, and it will sway (horizontally).

For small periodic vertical displacements $|a_{tt}|$, each instability interval contains a point $2g/L\omega^2 = k^2/4$ with $k$ an integer. This corresponds to a natural frequency of the ponytail which is half the frequency of the head. With $L = 25$ cm (the length of the modeled ponytail) the natural frequency ($\omega$) is $\approx 8.85$ rad./sec., which corresponds to 169 steps/min., which is within the typical jogging cadence of 170 to 190 steps/min. of an elite runner.

The next model represented the ponytail as a flexible string (inextensible flexible string of constant density) and included a linearized problem followed by a perturbation, with the only solution being regular when $s = L$ is a constant multiple of the Bessel function $J_0$ which yields the same situation as before except resulting in a value of 144 steps/min. which is more achievable by “average” runners.

Finally, the article concludes with the ponytail described as a flexible rod (inextensible flexible rod with a small bending stiffness), the most “realistic case” according to the author. This would lead to a singular perturbation problem, which could be solved using matched asymptotic expansions, or two-timing and a numerical implementation of Floquet theory. The author concludes with a remark that these methods would be interesting to apply to this problem.

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\textsuperscript{1}Keller, Joseph B., \textit{Ponytail Motion}, SIAM Journal on Applied Mathematics, Vol. 70, No. 7, pp. 2667-2672

\textsuperscript{2}Hill, G.W., American astronomer, studied the periodic motion of the Moon about Earth to determine its stability.


\textsuperscript{4}Mangus, W. and Winkler, S., \textit{Hill’s Equation}, Interscience, New York, 1966