4.7 Gaussian Quadrature

Motivation: When approximate $\int_a^b f(x)dx$, nodes $x_0, x_1, \cdots, x_n$ in $[a,b]$ are not equally spaced and result in the greatest degree of precision (accuracy).

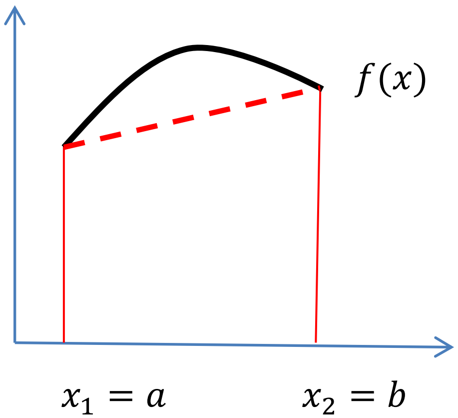

Figure 1 Trapezoidal rule

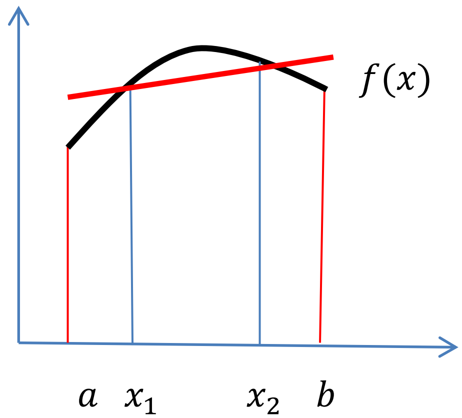

Figure 2 Gaussian quadrature

Consider $\int_a^b f(x)dx \approx \sum_{i=1}^n c_i f(x_i)$. Here $c_1, \cdots, c_n$ and $x_1, \cdots, x_n$ are $2n$ parameters. We therefore determine a class of polynomials of degree at most $2n-1$ for which the quadrature formulas have the degree of precision less than or equal to $2n-1$.

Example Consider $n = 2$ and $[a,b] = [-1,1]$. We want to determine x_1, x_2, c_1 and c_2 so that quadrature formula $\int_{-1}^1 f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$ has degree of precision 3.

Solution: Let $f(x) = 1$. $c_1 + c_2 = \int_{-1}^1 1dx = 2$ (Eq. 1) Let $f(x) = x$. $c_1x_1 + c_2x_2 = \int_{-1}^1 xdx = 0$ (Eq. 2)

Let $f(x) = x^2$. $c_1x_1^2 + c_2x_2^2 = \int_{-1}^1 x^2dx = \frac{2}{3}$ (Eq. 3) Let $f(x) = x^3$. $c_1x_1^3 + c_2x_2^3 = \int_{-1}^1 x^3dx = 1$ (Eq. 4)

Use equations (1)-(4) to solve for x_1, x_2, c_1 and c_2.

We obtain:

$$\int_{-1}^1 f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

Remark: Quadrature formula $\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$ has degree of precision 3. Trapezoidal rule has degree of precision 1.

Legendre Polynomials

Legendre polynomials satisfy: 1) For each n, $P_n(x)$ is a monic polynomial of degree n. 2) $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than n ($P(x)$ and $P_n(x)$ are orthogonal).
First five Legendre polynomials: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - 1/3$, $P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.

Theorem 4.7 Suppose that $x_1, \cdots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1,2, \cdots n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1;\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$$

Remark: Gaussian quadrature formula (more in Table 4.12)

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

n	Abscissae (x_i)	Weights (c_i)	Degree of Precision
2	$\sqrt{3}/3$	1.0	3
	$-\sqrt{3}/3$	1.0	
3	0. 7745966692	0.5555555556	5
	0.0	0.8888888889	
	-0.7745966692	0.5555555556	

Example 1 Approximate $\int_{-1}^{1} e^x \cos(x) dx$ using Gaussian quadrature with n = 3.

Gaussian quadrature on arbitrary intervals

Use substitution or transformation to transform $\int_a^b f(x) dx$ into an integral defined over $[-1,1]$.
Let $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$, with $t \in [-1, 1]$
Then

$$\int_a^b f(x) dx = \int_{-1}^{1} f\left(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\right)\frac{b-a}{2} dt$$

Example 2 Consider $\int_1^3 x^6 - x^2 \sin(2x)\, dx = 317.3442466$. Compare results from the closed Newton-Cotes formula with n=1, the open Newton-Cotes formula with n =1 and Gaussian quadrature when n = 2.
Solution:

(a) n = 1 closed Newton-Cotes formula (Trapezoidal rule): $\int_1^3 x^6 - x^2 \sin(2x)\, dx \approx \frac{2}{2}[f(1) + f(3)] = 731.605$
(b) n = 1 open Newton-Cotes formula: $\int_1^3 x^6 - x^2 \sin(2x)\, dx \approx \frac{3}{2}\left(\frac{2}{3}\right)\left[f\left(\frac{5}{3}\right) + f\left(\frac{7}{3}\right)\right] = 188.786$
(c) n = 2 Gaussian quadrature:

$$\int_1^3 x^6 - x^2 \sin(2x)\, dx = \int_{-1}^{1} f\left(\frac{1}{2}(4) + \frac{1}{2}(2)t\right)\frac{2}{2} dt = \int_{-1}^{1} ((t+2)^6 - (t+2)^2 \sin(t+2)) dt$$
$$\approx \left(\left(\frac{\sqrt{3}}{3}+2\right)^6 - \left(\frac{\sqrt{3}}{3}+2\right)^2 \sin\left(\frac{\sqrt{3}}{3}+2\right)\right)(1) + \left(\left(\frac{-\sqrt{3}}{3}+2\right)^6 - \left(\frac{-\sqrt{3}}{3}+2\right)^2 \sin\left(\frac{-\sqrt{3}}{3}+2\right)\right)(1)$$
$$= 306.820$$