Doolittle's method of LU factorization

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

By matrix-matrix multiplication

$$a_{1j} = u_{1j}, \qquad j = 1,2,\dots,n$$

$$a_{ij} = \begin{cases} \sum_{t=1}^{j} l_{it}u_{tj}, & \text{when } j < i \\ \sum_{t=1}^{i-1} l_{it}u_{tj} + u_{ij} & \text{when } j \ge i \end{cases}$$

Therefore

$$u_{1j} = a_{1j}, \qquad j = 1,2,\dots,n \ (1st \text{ row of } U)$$

$$l_{j1} = a_{j1}/u_{11}, \qquad j = 1,2,\dots,n \ (1st \text{ column of } L)$$

For $i = 2,3,\dots,n-1$ **do**

$$u_{ii} = a_{ii} - \sum_{t=1}^{i-1} l_{it}u_{tj}$$

$$u_{ij} = a_{ij} - \sum_{t=1}^{i-1} l_{it}u_{tj} \qquad \text{for } j = i+1,\dots,n \ \ (ith \text{ row of } U)$$

$$l_{ji} = \frac{a_{ji} - \sum_{t=1}^{i-1} l_{jt}u_{ti}}{u_{ii}} \qquad \text{for } j = i+1,\dots,n \ \ (ith \text{ column of } L)$$

End

$u_{nn} = a_{nn} - \sum_{t=1}^{n-1} l_{nt}u_{tn}$

LDL^t factorization for positive definite matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = LDL^t = \begin{bmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} \begin{bmatrix} 1 & l_{21} & \dots & l_{n1} \\ 0 & 1 & \dots & l_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

For $i = 1,2,\dots,n$ **do**

$$d_i = a_{ii} - \sum_{t=1}^{i-1} l_{it} l_{it} d_t$$

$$l_{ji} = \frac{a_{ji} - \sum_{t=1}^{i-1} l_{jt} l_{it} d_t}{d_i}$$ for $j = i+1,\dots,n$ (ith column of L)

End

Cholesky factorization (LL^t) for positive definite matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = LL^t = \begin{bmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \dots & l_{n1} \\ 0 & l_{22} & \dots & l_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_{nn} \end{bmatrix}$$

By matrix-matrix multiplication

$$l_{11}=\sqrt{a_{11}}$$

$$l_{j1}=\frac{a_{j1}}{l_{11}} \qquad \text{for } j=2,\dots,n \ (1st\ column\ of\ L)$$

For $i=2,3,\dots,n-1$ do

$$l_{ii}=\left(a_{ii}-\sum_{k=1}^{i-1} l_{ik}^2\right)^{\frac{1}{2}}$$

For $j=i+1,\dots,n$ do

$$l_{ji}=\left(a_{ji}-\sum_{k=1}^{i-1} l_{jk}l_{ik}\right)/l_{ii}$$

End

End

$$l_{nn}=\left(a_{nn}-\sum_{k=1}^{i-1} l_{nk}^2\right)^{\frac{1}{2}}$$

Crout factorization for Tridiagonal matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & \dots & 0 \\ a_{21} & a_{22} & a_{23} & & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \dots & \dots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix} = LU = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & l_{n,n-1} & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & \dots & 0 \\ 0 & 1 & u_{23} & & \vdots \\ 0 & \ddots & \ddots & \ddots & u_{n-1,n} \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$

$$a_{11} = l_{11}$$

$$a_{i,i-1} = l_{i,i-1} \qquad \text{for } i = 2,3,\dots,n$$

$$a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii} \qquad \text{for } i = 2,3,\dots,n$$

$$a_{i,i+1} = l_{ii}u_{i,i+1} \qquad \text{for } i = 1,2,\dots,n-1$$

Algorithm.

$l_{11} = a_{11}$

$u_{12} = a_{12}/l_{11}$

For $i = 1,2,\dots,n-1$ **do**

$l_{i,i-1} = a_{i,i-1}$

$l_{ii} = a_{ii} - l_{i,i-1}u_{i-1,i}$

$u_{i,i+1} = a_{i,i+1}/l_{ii}$

End

$l_{n,n-1} = a_{n,n-1}$

$l_{nn} = a_{nn} - l_{n,n-1}u_{n-1,n}$