

1.1 Review of Calculus

Limits and Continuity

Definition

A function f defined on a set X of real numbers has the *limit* L at x_0 , written $\lim_{x \rightarrow x_0} f(x) = L$, if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \quad \text{whenever} \quad x \in X \text{ and } 0 < |x - x_0| < \delta.$$

Definition

Let f be a function defined on a set X of real numbers and $x_0 \in X$. Then f is *continuous* at x_0 if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

The function f is continuous on the set X if it is continuous at each number in X .

Limits of Sequence

Definition

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real or complex numbers. The sequence $\{x_n\}_{n=1}^{\infty}$ has the *limit* x is, for any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $|x_n - x| < \varepsilon$, whenever $n > N(\varepsilon)$. The notation

$$\lim_{n \rightarrow \infty} x_n = x, \text{ or } x_n \rightarrow x \text{ as } n \rightarrow \infty,$$

means that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x .

Theorem

If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

- 1 f is continuous at x_0 ;
- 2 If the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to x_0 , then $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

Derivative

Definition

Let f be a function defined in an open interval containing x_0 . The function f is *differentiable* at x_0 if

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the *derivative* of f at x_0 . A function that has a derivative at each number in a set X is *differentiable on X* .

• Theorem

If the function f is differentiable at x_0 , then f is continuous at x_0 .

- **Rolle's Theorem**

Suppose $f \in C[a, b]$ and f is differentiable on (a, b) . If $f(a) = f(b)$, then a number c in (a, b) exists with $f'(c) = 0$.

- **Mean Value Theorem**

If $f \in C[a, b]$ and f is differentiable on (a, b) , then a number c in (a, b) exists with $f'(c) = \frac{f(b) - f(a)}{b - a}$.

- **Extreme Value Theorem**

If $f \in C[a, b]$, then $c_1, c_2 \in [a, b]$ exist with $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a, b]$. Additionally, if f is differentiable on (a, b) , then the numbers c_1 and c_2 occur either at the endpoints of $[a, b]$ or where f' is zero.

Taylor's Theorem

Suppose $f \in C^n[a, b]$, that $f^{(n+1)}$ exists on $[a, b]$ and $x_0 \in [a, b]$. For every $x \in [a, b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

Where

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots \\ + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

- $P_n(x)$ – nth Taylor polynomial
- $R_n(x)$ – remainder term (truncation error)
- When $x_0 = 0$, $P_n(x)$ is also called Maclaurin polynomial