11.1 -11.4 Boundary-Value Problems for Ordinary Differential Equations

11.1

Cooling Example: A beam of rectangular cross section subject to fixed temperature at two ends:

$$\frac{d^2T}{dx^2} - m^2T = 0 \ ,$$

$$m = \text{ heat transfer coefficient}$$

and, $T(x=0) = T_0$; $\quad T(x=L) = T_1$.

Theorem

Suppose the linear boundary-value problem (BVP)

$$y'' = p(x)y' + q(x)y + r(x), \text{ for } a \leq x \leq b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta \tag{1}$$

satisfies
(i) $p(x)$, $q(x)$ *and* $r(x)$ *are continuous on* $[a, b]$,
(ii) $q(x) > 0$ *on [a,b].*
Then the linear BVP has a unique solution.

Shooting Method for Linear BVP

Consider the following initial value problems:

$$y'' = p(x)y' + q(x)y + r(x) \text{ with } a \leq x \leq b, y(a) = \alpha, \text{ and } y'(a) = 0 \quad (2)$$

$$y'' = p(x)y' + q(x)y, \text{ with } a \leq x \leq b, y(a) = 0, \text{ and } y'(a) = 1 \ . \quad (3)$$

Let $y_1(x)$ be the solution to Eq. (2) and $y_2(x)$ be the solution to Eq. (3). Then

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x) \quad (4)$$

is the solution to the linear-BVP (1).

Solving IVP defined by Eq. (2)

Rewrite as two first order ODEs

$$\begin{aligned} \frac{dy}{dx} &= z(x) \\ \frac{dz}{dx} &= p(x)z + q(x)y + r(x) \end{aligned} \tag{5}$$

with $y(a) = \alpha$ and $z(a) = 0$.

Solving IVP defined by Eq. (3)

Rewrite as two first order ODEs

$$\begin{aligned} \frac{dy}{dx} &= w(x) \\ \frac{dw}{dx} &= p(x)w + q(x)y \end{aligned} \tag{6}$$

with $y(a) = 0$ and $w(a) = 1$.

RK4 for solving Eq. (5): Set $Y_0 = \alpha, Z_0 = 0$

for $(i = 0; i < N; i++)$

$$x_i = a + ih. \quad k_{1,Y} = hZ_i; \quad k_{1,Z} = h(p(x_i)Z_i + q(x_i)Y_i + r(x_i)); \tag{7}$$

$$k_{2,Y} = h(Z_i + 1/2k_{1,Z});$$

$$k_{2,Z} = h(p(x_i + \frac{h}{2})(Z_i + \frac{1}{2}k_{1,Z}) + q(x_i + \frac{h}{2})(Y_i + \frac{1}{2}k_{1,Y}) + r(x_i + \frac{h}{2})) \tag{8}$$

$$k_{3,Y} = h(Z_i + 1/2k_{2,Z});$$

$$k_{3,Z} = h(p(x_i + \frac{h}{2})(Z_i + \frac{1}{2}k_{2,Z}) + q(x_i + \frac{h}{2})(Y_i + \frac{1}{2}k_{2,Y}) + r(x_i + \frac{h}{2})) \tag{9}$$

$$\begin{aligned} k_{4,Y} &= h(Z_i + k_{3,Z}); \\ k_{4,Z} &= h(p(x_i + h)(Z_i + k_{3,Z}) + q(x_i + h)(Y_i + k_{3,Y}) + r(x_i + h)) \end{aligned} \tag{10}$$

$$\begin{aligned} Y_{i+1} &= Y_i + 1/6(k_{1,Y} + 2k_{2,Y} + 2k_{3,Y} + k_{4,Y}); \\ Z_{i+1} &= Z_i + 1/6(k_{1,Z} + 2k_{2,Z} + 2k_{3,Z} + k_{4,Z}) \end{aligned} \tag{11}$$

Shooting Method for Nonlinear BVP

Consider to solve

$$y'' = f(x, y, y'), \quad \text{for } a \le x \le b, \text{ with } y(a) = \alpha \text{ and } y(b) = \beta \ . \tag{12}$$

Example: $-\frac{d}{dx}\left(\kappa(y)\frac{dy}{dx}\right) + y(x) = p(x), \ 0 < x < 1$

with diffusion coefficient $\kappa(y) = \frac{1}{\sqrt{(dy/dx)^2 + c}}$

11.2 Basic idea of shooting method for nonlinear BVP

Construct a sequence of IVPs with parameter t, which have the form

$$y'' = f(x, y, y'), \quad \text{for } a \le x \le b, \ \text{with } y(a) = \alpha \text{ and } y'(a) = t \ . \tag{13}$$

Choosing parameters $t = t_k$ to ensure that

$$\lim_{k->\infty} y(b, t_k) = y(b) = \beta \ . \tag{14}$$

Secant method

Recast Eq. (14) as a root-finding problem:

$$y(b, t) - \beta = 0 \ . \tag{15}$$

Solving Eq. (15) by Secant method:

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)\,(t_{k-1} - t_{k-2})}{y(b, t_{k-1}) - y(b, t_{k-2})} \ , \qquad k = 2, 3, \cdots \tag{16}$$

Newton's method (1)

Solving Eq. (15) by Newton's method:

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)}{dy(b, t_{k-1})/dt}\ , \qquad k = 1, 3, \cdots \tag{17}$$

Differentiating Eq. (13) $y''(x,t) = f(x, y(x,t), y'(x,t))$ with respect to t:

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial t} \tag{18}$$

Differentiating initial condition of Eq. (13) gives: $\frac{\partial y}{\partial t} = 0; \quad \frac{\partial y'}{\partial t} = 1.$
Let $z(x,t) = \frac{\partial y}{\partial t}(x,t)$ and assume orders of differentiation of x and t can be reversed.
Eq. (18) becomes an IVP:

$$z''(x,t) = \frac{\partial f}{\partial y}z + \frac{\partial f}{\partial y'}z'(x,t) \text{ with } z(a,t) = 0; \text{ and } z'(a,t) = 1. \tag{19}$$

Newton's method (2)

Eq. (17) can be solved by

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)}{z(b, t_{k-1})} \ , \qquad k = 1, 3, \cdots \tag{20}$$

Nonlinear shooting method - Newton's method

Let M be max. number of iterations. Set $TK = \frac{\beta-\alpha}{b-a}$; $\quad k = 1$; $\quad t_0 = TK$.
while $(k \leq M)$ **do**
(i) Solve the following IVPs respectively:

$$\begin{aligned} y'' &= f(x, y, y'), \text{ for } a \leq x \leq b, \text{ with } y(a) = \alpha \text{ and } y'(a) = t_{k-1} \ ; \\ z''(x, t_{k-1}) &= \frac{\partial f}{\partial y} z(x, t_{k-1}) + \frac{\partial f}{\partial y'} z'(x, t_{k-1}) \text{ for } a \leq x \leq b \\ &\text{with } z(a, t_{k-1}) = 0; \text{ and } z'(a, t_{k-1}) = 1. \end{aligned} \tag{21}$$

(ii) Solve for t_k:

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)}{z(b, t_{k-1})} \ , \tag{22}$$

(iii) If $|y(b, t_k) - \beta| < \varepsilon$, **STOP**.

Example of nonlinear shooting method

Solve $y'' = 1/8(32 + 2x^3 - yy')$, for $1 \le x \le 3$, with $y(1) = 17$ and $y(3) = 43/3$.

Solution:

$t_0 = (43/3 - 17)/(3 - 1)$

$$y'' = 1/8(32 + 2x^3 - yy'), \quad \text{with } y(1) = 17, \text{ and } y'(1) = t_{k-1} \ .$$

$$z'' = \frac{\partial f}{\partial y} z(x, t_{k-1}) + \frac{\partial f}{\partial y'} z'(x, t_{k-1}) = -\frac{1}{8}(y'z + yz')$$

with $z(1, t_{k-1}) = 0;$ and $z'(1, t_{k-1}) = 1.$

Consider to solve Eq. (12)
(i) divide interval $[a, b]$ into N subintervals with $h = (b-a)/N$, $x_i = a + ih$ for $i = 0, \cdots, N+1$.
(ii) approximate derivatives of Eq. (12) by differences respectively:

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = f\left(x_i, y(x_i), \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - h^2/6y'''(\eta_i)\right) + h^2/12y^{(4)}(\xi_i). \tag{23}$$

Finite difference method:
Set $w_0 = \alpha$, $w_{N+1} = \beta$.

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0 \ , \quad \text{for each } i = 1, 2, \cdots, N \ . \tag{24}$$

Individual equations of (24) are:

$$
\begin{aligned}
w_2 - 2w_1 - h^2 f\left(x_1, w_1, \frac{w_2 - \alpha}{2h}\right) + \alpha = 0 \ , \\
w_{i+1} - 2w_i + w_{i-1} - h^2 f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0 \ , \\
\text{for each } i = 2, 3, \cdots, N-1 \ . \\
-2w_N + w_{N-1} - h^2 f\left(x_N, w_N, \frac{\beta - w_{N-1}}{2h}\right) + \beta = 0 \ .
\end{aligned}
\tag{25}
$$

Solve Eq. (25) for $w_1, w_2, \cdots, w_N$ by Newton's method.