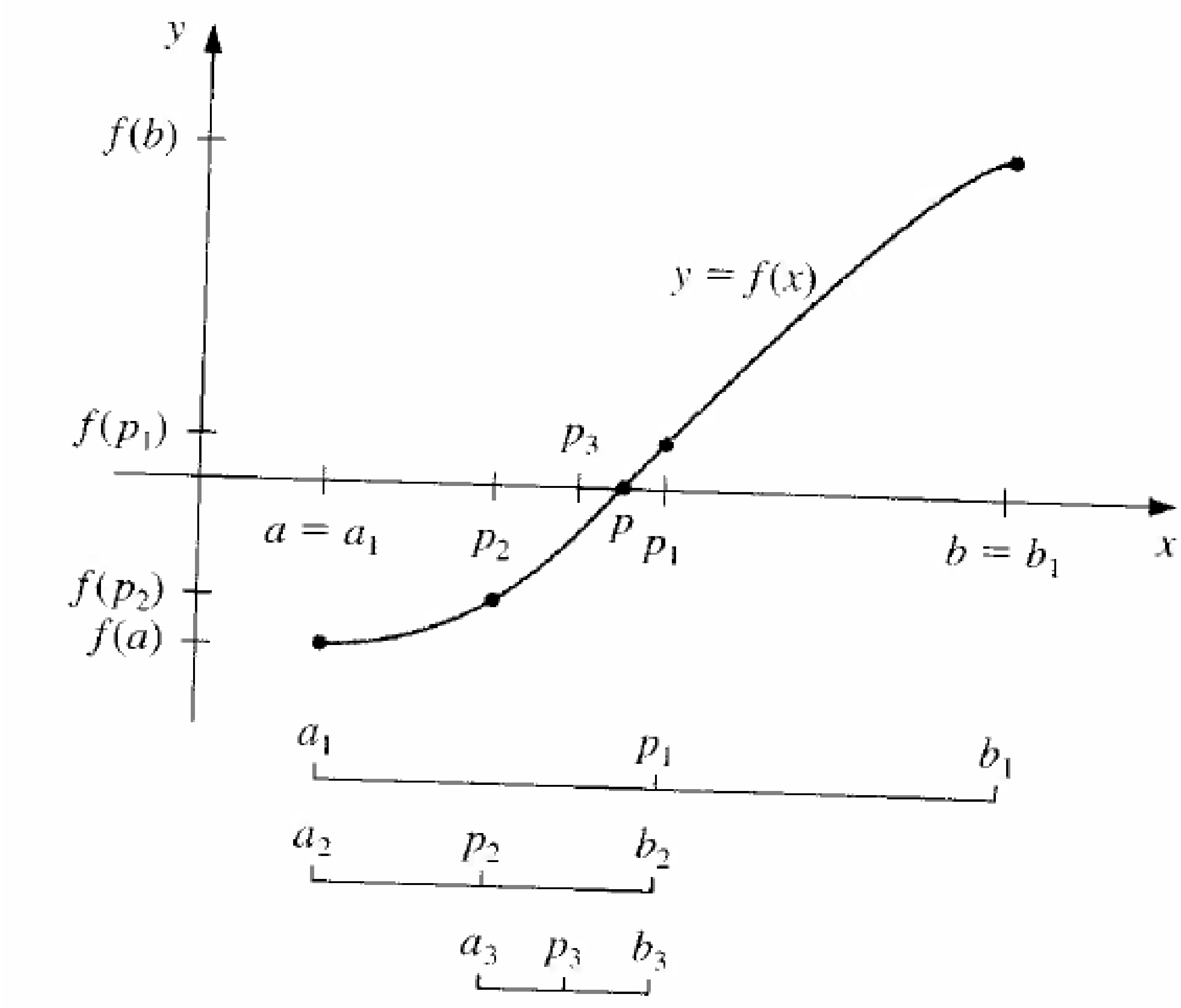
2.1 The Bisection Method

Basic Idea

- Suppose function $f(x)$ is continuous on $[a, b]$, and $f(a)$, $f(b)$ have opposite signs.
- By the Intermediate Value Theorem (IVT), there must exist an p in (a, b) with $f(p) = 0$.
- Bisect (sub)intervals and apply IVT repeatedly.



- The sequence of intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ contains the desired root.

- Intervals containing the root: $(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset (a_4, b_4) \ldots$

- After n steps, the interval (a_n, b_n) has the length:

$$b_n - a_n = \left(1/2\right)^{n-1}(b - a)$$

- Let $p_n = \frac{b_n + a_n}{2}$ be the mid-point of (a_n, b_n). The limit of sequence $\{p_n\}_{n=1}^{\infty}$ is the root.

The Algorithm

```
INPUT       a,b; tolerance TOL; maximum number of iterations N0.
OUTPUT      solution p or message of failure.
STEP1       Set  i = 1;
            FA = f(a);
STEP2       While i ≤ N0 do STEPs 3-6.
            STEP3 Set p = a + (b-a)/2;     // a good way of computing middle point
                  FP = f(p).
            STEP4  IF FP = 0 or (b-a) < TOL then
                      OUTPUT (p);
                      STOP.
            STEP5     Set i = i +1.
            STEP6     If FP·FA > 0 then
                        Set a = p;
                        FA = FP.
                      else
                        set b = p;
STEP7 OUTPUT("Method failed after N0 iterations");
      STOP.
```

Matlab Code

```
function p=bisection(f,a,b,tol)

while 1
    p=(a+b)/2;
    if p-a<tol, break; end
    if f(a)*f(p)>0
        a=p;
    else
        b=p;
    end
end   %while 1
```

Stopping Criteria

- Method 1: $|p_n - p_{n-1}| < \varepsilon$

- Method 2: $\begin{cases} \frac{|p_n - p_{n-1}|}{|p_n|} < \varepsilon, \quad p_n \neq 0 \ \text{ or} \\ \qquad\qquad |f(p_n)| < \varepsilon \end{cases}$

- None is perfect. Use a combination in real applications.

Convergence

- **Theorem**

Suppose function $f(x)$ is continuous on $[a, b]$, and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of $f(x)$ with

$$|p_n - p| = \left({}^{1}/_{2}\right)^n (b - a), \qquad \text{when } n \geq 1$$

- **Convergence rate**

The sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with the rate of convergence $O\left(\left({}^{1}/_{2}\right)^n\right)$:

$$p_n = p + O\left(\left({}^{1}/_{2}\right)^n\right)$$