# 2.2 Fixed-Point Iteration

# Basic Definitions

- A number $p$ is a ***fixed point*** for a given function $g$ if $g(p) = p$
- Root finding $f(x) = 0$ is related to fixed-point iteration $g(p) = p$
  - Given a root-finding problem $f(p) = 0$, there are many $g$ with fixed points at $p$:

  Example: $g(x) \coloneqq x - f(x)$

  $$g(x) \coloneqq x + 3f(x)$$

  …

  If $g$ has fixed point at $p$, then $f(x) = x - g(x)$ has a zero at $p$

# Why study fixed-point iteration?

1. Sometimes easier to analyze
2. Analyzing fixed-point problem can help us find good root-finding methods

**A Fixed-Point Problem**

Determine the fixed points of the function $g(x) = x^2 - 2$.

# When Does Fixed-Point Iteration Converge?

**Existence and Uniqueness Theorem**

a. If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then $g$ has a fixed-point in $[a,b]$

b. If, in addition, $g'(x)$ exists on $(a,b)$ and a positive constant $k < 1$ exists with

$$|g'(x)| \le k, \qquad \text{for all } x \in (a,b),$$

then there is **exactly one** fixed point in $[a,b]$.

**Note:**

*1.* $g \in C[a,b] -$ $g$ is continuous in $[a,b]$

*2.* $g(x) \in C[a,b] - g$ takes values in $[a,b]$

# Proof

- If $g(a) = a$, or $g(b) = b$, then $g$ has a fixed point at the endpoint.
- Otherwise, $g(a) > a$ and $g(b) < b$.
- Define a new function $h(x) = g(x) - x$
  - $h(a) = g(a) - a > 0$ and $h(b) = g(b) - b < 0$
  - $h$ is continuous
- By intermediate value theorem, there exists $p \in (a, b)$ for which $h(p) = p$.
  - Thus, $g(p) = p$.

b) $|g'(x)| \leq k < 1$. Suppose we have two fixed points $p$ and $q$.

By the mean value theorem, there is a number $\xi$ between $p$ and $q$ with

$$g'(\xi) = \frac{g(p) - g(q)}{p - q}$$

Thus $|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \leq k|p - q| < |p - q|$, which is a contradiction.

This shows the supposition is false. Hence, the fixed point is unique.

# Fixed-Point Iteration

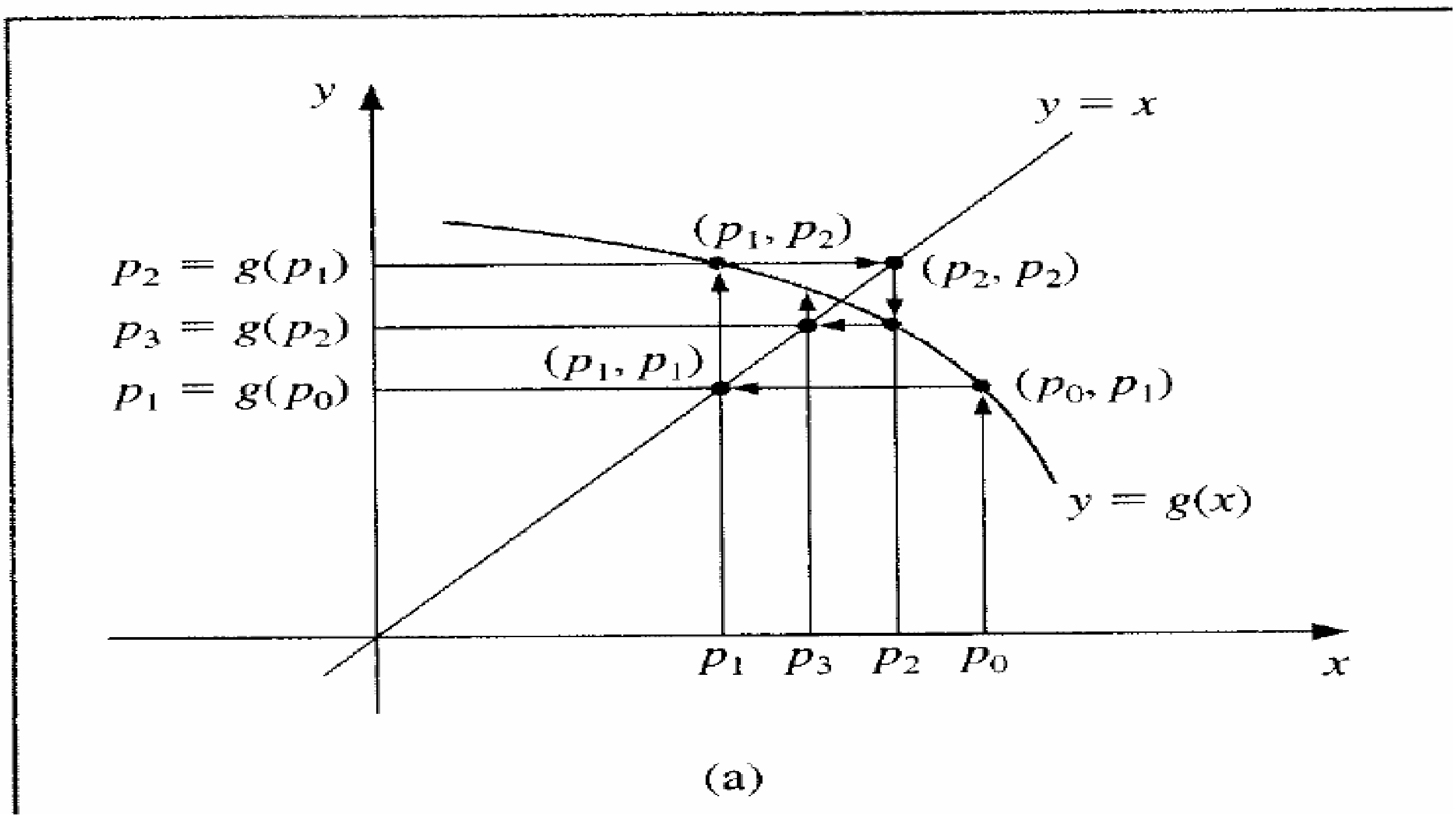
- For initial $p_0$, generate sequence $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$.
- If the sequence converges to $p$, then
$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

**A Fixed-Point Problem**

Determine the fixed points of the function $g(x) = \cos(x)$ for $x \in [-0.1, 1.8]$.

Remark: See also the Matlab code.

# The Algorithm




(a)

```
INPUT     p0; tolerance TOL; maximum number of iteration N0.
OUTPUT  solution p or message of failure
STEP1     Set i = 1.                                   // init. counter
STEP2     While i ≤ N0 do Steps 3-6
          STEP3     Set p = g(p0).
          STEP4     If |p-p0| < TOL  then
                       OUTPUT(p);                // successfully found the solution
                       STOP.
          STEP5     Set i = i + 1.
          STEP6     Set p0 = p.                  // update p0
STEP7     OUTPUT("The method failed after N0 iterations");
           STOP.
```

# Convergence

**Fixed-Point Theorem**

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, that $g'$ exists on $(a, b)$ and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \qquad \text{for all } x \in (a, b)$$

Then, for any number $p_0$ in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1})$$

converges to the unique fixed point $p$ in $[a, b]$.

**Corollary**

If $g$ satisfies the above hypotheses, then bounds for the error involved using $p_n$ to approximating $p$ are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

$$|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p_0|$$

Proof: $|p_n - p| = |g(p_{n-1}) - g(p)|$

$$= |g'(\xi_n)||p_{n-1} - p| \quad by\ MVT$$

$$\leq k|p_{n-1} - p|$$

Remark: Since $k < 1$, the distance to fixed point is shrinking every iteration

Keep doing the above procedure:

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots$$

$$\leq k^n|p_0 - p|.$$

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p| = 0$$