2.5 Accelerating Convergence

Aitken's Δ^2 Method

- **Assume** $\{p_n\}_{n=0}^{\infty}$ is a linearly convergent sequence with limit p.
- Further assume $\frac{|p_{n+1}-p|}{|p_n-p|} \approx \frac{|p_{n+2}-p|}{|p_{n+1}-p|}$ when n is large
- Solving for p yields:

$$p \approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

A little algebraic manipulation gives:

$$p \approx p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

- Define $\widehat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

Remark: The new sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster.

Definition

Aitken's Δ^2 Method: Given a sequence $\{p_n\}_{n=0}^{\infty}$ which converges to limit p. The new sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ defined by $\widehat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$ converges more rapidly to p than does the sequence $\{p_n\}_{n=0}^{\infty}$.

Remark:

1. numerator $p_{n+1} - p_n$ is a forward difference
2. denominator $p_{n+2} - 2p_{n+1} + p_n$ is central difference.

Example. Consider the sequence $\{p_n\}_{n=0}^{\infty}$ generated by the fixed point iteration $p_{n+1} = \cos(p_n), p_0 = 0.$

iteration	p_n	$\widehat{p_n}$
0	0.000000000000000	0.685073357326045
1	1.000000000000000	0.728010361467617
2	0.540302305868140	0.733665164585231
3	0.857553215846393	0.736906294340474
4	0.654289790497779	0.738050421371664
5	0.793480358742566	0.738636096881655
6	0.701368773622757	0.738876582817136
7	0.763959682900654	0.738992243027034
8	0.722102425026708	0.739042511328159
9	0.750417761763761	0.739065949599941
10	0.731404042422510	0.739076383318956
11	0.744237354900557	0.739081177259563*
12	0.735604740436347	0.739083333909684*

Remark: $\widehat{p_{11}}$ needs p_{13}; $\widehat{p_{12}}$ needs p_{14}. p_{13} and p_{14} are not shown here.

Theorem. Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges linearly to the limit p and that

$$\lim_{n\to\infty} \frac{p_{n+1} - p}{p_n - p} < 1$$

The Aitken's Δ^2 sequence $\{\widehat{p_n}\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\widehat{p_n} - p}{p_n - p} = 0$$

Steffensen's Method

- Steffensen's Method is a combination of fixed-point iteration and the Aitken's Δ^2 method:

  Suppose we have a fixed point iteration:

  $$p_0, \quad p_1 = g(p_0), \quad p_2 = g(p_1), \quad \dots$$

  Once we have p_0, p_1 and p_2, we can compute

  $$\hat{p}_0 = p_0 - \frac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0}$$

  At this point we "restart" the fixed point iteration with $p_0 = \hat{p}_0$, *e.g.*

  $$p_3 = \hat{p}_0, \quad p_4 = g(p_3), \quad p_5 = g(p_4),$$

  and compute

  $$\hat{p}_3 = p_3 - \frac{(p_4 - p_3)^2}{p_5 - 2p_4 + p_3}$$

Example. Compare Fixed point iteration, Newton's method and Steffensen's method for solving:

$$f(x) = x^3 + 4x^2 - 10 = 0.$$

Solution:

1. Fixed point iteration: $p_{n+1} = g(p_n) = \sqrt{\frac{10}{p_n+4}}$

i	p_n	$g(p_n)$
0	1.50000	1.34840
1	1.34840	1.36738
2	1.36738	1.36496
3	1.36496	1.3652
4	1.36526	1.36523
5	1.36523	1.36523

2. Newton's method

i	x_n	$f(x_n)$
0	1.50000	1.51600e-01
1	1.36495	-3.11226e-04
2	1.36523	-1.35587e-09

3. Steffensen's method

p_0	p_1	p_2	$\widehat{p_0}$	$\lvert p_2 - \widehat{p_0}\rvert$
1.50000	1.34840	1.36738	1.36527	3.96903e-05
p_3	p_4	p_5	$\widehat{p_3}$	$\lvert p_3 - \widehat{p_3}\rvert$
1.36527	1.36523	1.36523	1.36523	2.80531e-12