2.6 Zeros of Polynomials and Horner's Method

Zeros of Polynomials

- **Definition**: Degree of a Polynomial

A **polynomial of degree $\boldsymbol{n}$** has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, a_n \neq 0$$

Where the $a_i's$ are constants (either real or complex) called the **coefficients** of $P(x)$

- **Fundamental Theorem of Algebra**

If $P(x)$ is a polynomial of degree $n \geq 1$, with real or complex coefficients, $P(x) = 0$ has at least one root.

- **Corollary**

There exists unique constants $x_1, x_2, \dots, x_k$ and unique positive integers $m_1, m_2, \dots, m_k$ such that $\sum_{i=1}^{k} m_i = n$ and

$$P(x) = a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \dots (x - x_k)^{m_k}$$

Remark:

1. Collection of zeros is unique
2. m_i are multiplicities of the individual zeros
3. A polynomial of degree n has exactly n zeros, counting multiplicity.

- **Corollary**

Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \dots, x_k$ with $k > n$ are distinct numbers with $P(x_i) = Q(x_i)$ for all $i = 1,2, \dots, k$, then $P(x) = Q(x)$ for all values of x.

Remark:

If two polynomials of degree n agree at at least (n+1) points, then they must be the same.

Horner's Method

- Horner's method is a technique to evaluate polynomials quickly. Need n multiplications and n additions to evaluate $P(x_0)$.
- Assume $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Evaluate $P(x_0)$.

Let $b_n = a_n, b_k = a_k + b_{k+1} x_0,$

$$\text{for } k = (n-1), (n-2), \ldots, 1, 0$$

Then $b_0 = P(x_0)$.

- At the same time, we also computed the decomposition:

$$P(x) = (x - x_0) Q(x) + b_0,$$

Where $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$

Remark:

1. $P'(x_0) = Q(x_0)$, which can be computed by using Horner's method in (n-1) multiplications and (n-1) additions.
2. Horner's method is nested arithmetic

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ &= (a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1) x + a_0 \\ &= ((a_n x^{n-2} + a_{n-1} x^{n-3} + \cdots) x + a_1) x + a_0 \\ &= \underbrace{(\ldots((}_{n-1} \underbrace{a_n x + a_{n-1}}_{b_{n-1}}) x + \cdots) x + a_1) x + a_0 \end{aligned}$$

- Example. Use Horner's method to evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$.

Solution:

Step 1: $a_4 = 2, \;\; b_4 = 2$

Step 2: $a_3 = 0, \; b_3 = a_3 + b_4 x_0 = 0 + 2(-2) = -4$

Step 3: $a_2 = -3, \; b_2 = a_2 + b_3 x_0 = -3 + (-4)(-2) = 5$

Step 4: $a_1 = 3, \; b_1 = a_1 + b_2 x_0 = 3 + (5)(-2) = -7$

Step 5: $a_0 = -4, \; b_0 = a_0 + b_1 x_0 = -4 + (-7)(-2) = 10$

Thus $P(-2) = b_0 = 10$

Remark:

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 =$$
$$\Big(\big((a_4 x + a_3)x + a_2\big)x + a_1\Big)x + a_0$$