3.3 Divided Differences

Representing nth Lagrange Polynomial

- If $P_n(x)$ is the nth degree Lagrange interpolating polynomial that agrees with $f(x)$ at the points $\{x_0, x_1, \ldots, x_n\}$, express $P_n(x)$ in the form:
  $P_n(x) =$
  $a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots +$
  $a_n(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1})$
- ? How to find constants $a_0, \ldots, a_n$?

Divided Differences

- Zeroth divided difference:

$$f[x_i] = f(x_i)$$

- First divided difference:

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

- Second divided difference:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

- kth divided difference:

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

- **Finding constants** $a_0, \ldots, a_n$.

1. $x = x_0$: $a_0 = P_n(x_0) = f(x_0) = f[x_0]$
2. $x = x_1$: $f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1)$

$$\Longrightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

3. In general: $a_k = f[x_0, x_1, \ldots, x_k]$ for $k = 0, \ldots, n$

Newton's Interpolatory Divided Difference Formula

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

Or

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} [f[x_0, \dots, x_k](x - x_0) \dots (x - x_{k-1})]$$

Table for Computing

x	f(x)	1st Div. Diff.	2nd Div. Diff.	
x_0	$f[x_0]$			
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	...
		$f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2}$		
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3, x_4] = \frac{f[x_4]-f[x_3]}{x_4-x_3}$		
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \frac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4, x_5] = \frac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Algorithm: Newton's Divided Differences

Input: $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$

Output: Divided differences $F_{0,0}, \dots, F_{n,n}$

//comment: $P_n(x) = F_{0,0} + \sum_{i=1}^{n}[F_{i,i}(x - x_0) \dots (x - x_{i-1})]$

Step 1: For $i = 0, \dots, n$

set $F_{i,0} = f(x_i)$

Step 2: For $i = 1, \dots, n$

For $j = 1, \dots, i$

set $F_{i,j} = \frac{F_{i,j-1} - F_{i-1,j-1}}{x_i - x_{i-j}}$

End

End

Output($F_{0,0}$,…$F_{i,i}$,…,$F_{n,n}$)

STOP.

Theorem. Suppose that $f \in C^n[a, b]$ and $x_0, x_1, \dots, x_n$ are distinct numbers in $[a, b]$. Then $\exists \xi \in (a, b)$ with $f[x_0, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}$.

Remark: When $n = 1$, it's just the Mean Value Theorem.

Forward difference formula for equally spaced nodes

- Let the points $\{x_0, x_1, \dots, x_n\}$ be equally spaced. $h = x_{i+1} - x_i$, for each $i = 0, \dots, n-1$;

  and $x = x_0 + sh$.
- Then

$$
\begin{aligned}
P_n(x) = f[x_0] &+ f[x_0, x_1](x - x_0) \\
&+ f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots \\
&+ f[x_0, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1}) \\
&= f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2] + \cdots \\
&+ s(s-1) \dots (s-n+1)h^n f[x_0, \dots, x_n]
\end{aligned}
$$

Or

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k f[x_0, x_1, \dots, x_k]$$

Where $\binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{k!}$

Example. 1) Complete the following divided difference table. 2) Find the interpolating polynomial.

i	x_i	$f[x_i]$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},\dots,x_i]$	$f[x_{i-4},\dots,x_i]$
0	1.0	0.7651977	-0.4837057			
1	1.3	0.6200860	-0.5489460			
2	1.6	0.4554022		-0.0494433		
3	1.9					
4	2.2	0.1103623				