4.7 Gaussian Quadrature

Gaussian quadrature is more accurate than the Newton-Cotes quadrature in the following sense:

1. Both Gaussian quadrature and Newton-Cotes quadrature use the similar idea to do the approximation, i.e. they both use the Lagrange interpolation polynomial to approximate the integrand function and integrate the Lagrange interpolation polynomial to approximate the given definite integral.

2. When the same number of nodes is used, the algebraic degree of precision of the Gaussian quadrature is higher than that of the Newton-Cotes quadrature.

Motivation: When approximate $\int_a^b f(x)dx$, nodes $x_0, x_1, \cdots, x_n$ in $[a,b]$ are not equally spaced and result in the greatest degree of precision (accuracy).

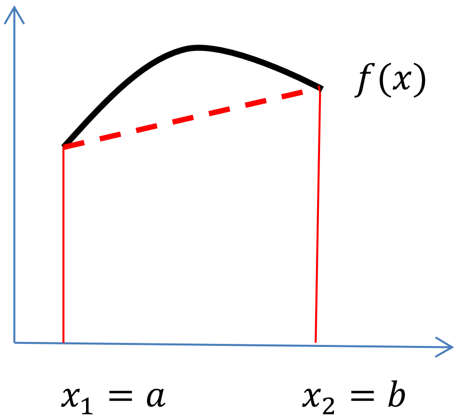

Figure 1 Trapezoidal rule

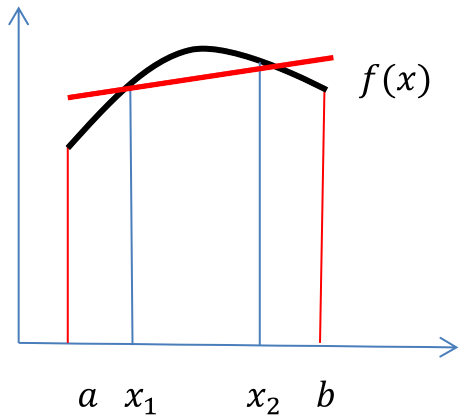

Figure 2 Gaussian quadrature

Consider $\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$. Here $c_1, \cdots, c_n$ and $x_1, \cdots, x_n$ are $2n$ parameters. We therefore can a determine quadrature formula that has the degree of precision $2n-1$ for which it is exact when integrating polynomials of degree at most $2n-1$.

Example Consider $n = 2$ and $[a,b] = [-1,1]$. We want to determine x_1, x_2, c_1 and c_2 so that quadrature formula $\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$ has degree of precision 3.

Solution: Let $f(x) = 1$. $c_1 + c_2 = \int_{-1}^{1} 1dx = 2$ (Eq. 1) Let $f(x) = x$. $c_1 x_1 + c_2 x_2 = \int_{-1}^{1} xdx = 0$ (Eq. 2)

Let $f(x) = x^2$. $c_1 x_1^2 + c_2 x_2^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$ (Eq. 3) Let $f(x) = x^3$. $c_1 x_1^3 + c_2 x_2^3 = \int_{-1}^{1} x^3 dx = 1$ (Eq. 4)

Use equations (1)-(4) to solve for x_1, x_2, c_1 and c_2.

We obtain:

$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

Remark: Quadrature formula $\int_{-1}^{1} f(x)dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$ has degree of precision 3. Trapezoidal rule has degree of precision 1.

Legendre Polynomials

Legendre polynomials satisfy: 1) For each n, $P_n(x)$ is a monic polynomial of degree n. 2) $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than n ($P(x)$ and $P_n(x)$ are orthogonal).
First five Legendre polynomials: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - 1/3$, $P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.

Theorem 4.7 Suppose that $x_1, \cdots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1,2,\cdots n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1;\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} dx$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$$

Remark: Gaussian quadrature formula (more in Table 4.12)

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

n	Abscissae (x_i)	Weights (c_i)	Degree of Precision
2	$\sqrt{3}/3$	1.0	3
	$-\sqrt{3}/3$	1.0	
3	0. 7745966692	0.5555555556	5
	0.0	0.8888888889	
	-0.7745966692	0.5555555556	

Example 1 Approximate $\int_{-1}^{1} e^x \cos(x)dx$ using Gaussian quadrature with n = 3.

Gaussian quadrature on arbitrary intervals

Use substitution or transformation to transform $\int_a^b f(x)dx$ into an integral defined over $[-1,1]$.
Let $x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$, with $t \in [-1,1]$
Then

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\right)\frac{b-a}{2}dt$$

Example 2 Consider $\int_1^3 x^6 - x^2\sin(2x)\,dx = 317.3442466$. Compare results from the closed Newton-Cotes formula with n=1, the open Newton-Cotes formula with n =1 and Gaussian quadrature when n = 2.
Solution:

(a) n = 1 closed Newton-Cotes formula (Trapezoidal rule): $\int_1^3 x^6 - x^2\sin(2x)\,dx \approx \frac{2}{2}[f(1) + f(3)] = 731.605$

(b) n = 1 open Newton-Cotes formula: $\int_1^3 x^6 - x^2\sin(2x)\,dx \approx \frac{3}{2}\left(\frac{2}{3}\right)\left[f\left(\frac{5}{3}\right) + f\left(\frac{7}{3}\right)\right] = 188.786$

(c) n = 2 Gaussian quadrature:

$$\int_1^3 x^6 - x^2\sin(2x)\,dx = \int_{-1}^{1} f\left(\frac{1}{2}(4) + \frac{1}{2}(2)t\right)\frac{2}{2}dt = \int_{-1}^{1}((t+2)^6 - (t+2)^2\sin(t+2))dt$$

$$\approx \left(\left(\frac{\sqrt{3}}{3}+2\right)^6 - \left(\frac{\sqrt{3}}{3}+2\right)^2\sin\left(\frac{\sqrt{3}}{3}+2\right)\right)(1) + \left(\left(\frac{-\sqrt{3}}{3}+2\right)^6 - \left(\frac{-\sqrt{3}}{3}+2\right)^2\sin\left(\frac{-\sqrt{3}}{3}+2\right)\right)(1)$$

$$= 306.820$$