5.1 Elementary Theory of Initial-Value Problems

Definition: A function $f(t,y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset R^2$ if a constant $L > 0$ exists with

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

whenever (t,y_1) and (t,y_2) are in D. The constant L is called a **Lipschitz constant** for f.

Example. Show that $f(t,y) = \frac{2}{t}y + t^2e^t$ satisfies a Lipschitz condition on the interval $D = \{(t,y)|1 \le t \le 2 \; and -2 \le y \le 5\}$.

Solution: For arbitrary points (t,y_1) and (t,y_2) in D, we have

$$|f(t,y_1) - f(t,y_2)| = \left|\left(\frac{2}{t}y_1 + t^2e^t\right) - \left(\frac{2}{t}y_2 + t^2e^t\right)\right| = |\frac{2}{t}||y_1 - y_2| \le 2|y_1 - y_2|$$

Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant $L = 2$.

Definition: A set $D \subset R^2$ is said to be **convex** if whenever (t_1,y_1) and (t_2,y_2) belongs to D and $\lambda \in [0,1]$, the point $((1-\lambda)t_1 + \lambda t_2, \; (1-\lambda)y_1 + \lambda y_2)$ also belongs to D.

Remark: 1. Convex means that line segment connecting (t_1,y_1) and (t_2,y_2) is in D whenever (t_1,y_1) and (t_2,y_2) belongs to D.

2. The set $D = \{(x,y) \; a \le t \le b \; and -\infty \le y \le \infty\}$ is convex.

Theorem 5.3 Suppose $f(t,y)$ is defined on a convex set $D \subset R^2$. If a constant $L > 0$ exists with

$$|\frac{\partial f}{\partial y}(t,y)| \le L$$

for all $(t,y) \in D$, then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

Existence and Uniqueness

Theorem 5.4 Suppose that $D = \{(x,y) \; a \le t \le b \; and -\infty < y < \infty\}$ and that $f(t,y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem (IVP)

$$y' = f(t,y), \quad a \le t \le b, \; y(a) = \beta,$$

has a unique solution $y(t)$ for $a \le t \le b$.

Example 2. Show that there is a unique solution to the IVP

$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \quad y(0) = 0.$$

Solution: $f(t, y) = 1 + t\sin(ty)$

Method 1. Use Mean Value Theorem in y, we have

$\frac{f(t,y_2)-f(t,y_1)}{y_2-y_1} = \frac{\partial f}{\partial y}(t, \xi) = t^2\cos(\xi t)$ for ξ in (y_1, y_2).

Thus, $|f(t, y_2) - f(t, y_1)| = |y_2 - y_1||t^2 \cos(\xi t)| \le 2^2|y_2 - y_1|$.

f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant $L = 4$.

Additionally, $f(t, y)$ is continuous on $\{0 \le t \le 2 \text{ and } -\infty < y < \infty\}$. **Thm 5.4** implies that a unique Solution exists.

Method 2. The set $\{0 \le t \le 2 \text{ and } -\infty < y < \infty\}$ is convex. $\left|\frac{\partial f}{\partial y}(t, y)\right| = |t^2\cos(yt)| \le 2^2(1) = 4$. So f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant $L = 4$.

Additionally, $f(t, y)$ is continuous on $\{0 \le t \le 2 \text{ and } -\infty < y < \infty\}$. **Thm 5.4** implies that a unique Solution exists.

Well-Posedness

Definition: The IVP $\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \beta$ is said to be **a well-posed problem** if:

1. A unique solution $y(t)$, to the problem exists, and
2. There exist constant $\varepsilon_0 > 0$ and $k > 0$ such that for any ε with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a, b]$, and when $|\delta_0| < \varepsilon$, the IVP (*a perturbed problem associated with original* $\frac{dy}{dt} = f(t, y)$)

$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \le t \le b, \quad z(a) = \beta + \delta_0$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \qquad \text{for all } t \text{ in } [a, b].$$

Why well-posedness? Numerical methods always solve perturbed problem because of round-off errors.

Example. Consider the original problem $y' = y - t^2 + 1, \quad 0 \le t \le 2, \; y(0) = 0.5$. The solution is

$$y(t) = (t+1)^2 - 0.5e^t.$$

The associated perturbed problem is $z' = z - t^2 + 1 + \delta, \quad 0 \le t \le 2, \; z(0) = 0.5 + \delta_0$ with δ and δ_0 being constants. Assume $|\delta| < \varepsilon$ and $|\delta_0| < \varepsilon$. The solution is

$$z(t) = (t+1)^2 + (\delta + \delta_0 - 0.5)e^t - \delta.$$

$$|z(t) - y(t)| = |(\delta + \delta_0)e^t - \delta| \le |\delta + \delta_0|e^t + |\delta| \le (\varepsilon + \varepsilon)e^2 + \varepsilon = (2e^2 + 1)\varepsilon$$

So $k = 2e^2 + 1$.

Theorem 5.6 Suppose $D = \{(x, y) \;\; a \le t \le b \; and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y, then IVP

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta,$$

is well-posed.

Example. Show the IVP $y' = y - t^2 + 1, \quad 0 \le t \le 2, \; y(0) = 0.5$
is well-posed on $D = \{(x, y) \;\; 0 \le t \le 2 \; and - \infty < y < \infty\}$

Solution: $\left|\frac{\partial}{\partial y}(y - t^2 + 1)\right| = |1| = 1.$

Function $(y - t^2 + 1)$ satisfies Lipschitz condition with L =1. So **Theorem 5.6** implies the IVP is well posed.

5.2 Euler's Method

Algorithm description

Suppose a well-posed IVP is

$$y' = f(t, y), \quad a \le t \le b, \; y(a) = \beta.$$

Distribute mesh points equally throughout $[a, b]$:

$$t_i = a + ih, \qquad \text{for each } i = 0, 1, 2, \cdots, N.$$

The step size $h = \frac{b-a}{N} = t_{i+1} - t_i$.

We compute the approximate solution at time points t_i by:

$w_0 = \beta$

$$w_{i+1} = w_i + hf(t_i, w_i), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Difference Eq.

Here $w_i \approx y(t_i)$, namely, w_i is the approximate solution at time t_i.

Example. Solve $y' = y - t^2 + 1, \quad 0 \le t \le 2, \; y(0) = 0.5$ numerically with time step size $h = 0.5$.

Solution: $w_0 = y(0) = 0.5$

$$y(0.5) \approx w_1 = w_0 + h(w_0 - (t_0)^2 + 1) = 0.5 + 0.5(0.5 - 0 + 1) = 1.25$$
$$y(1.0) \approx w_2 = w_1 + h(w_1 - (t_1)^2 + 1) = 1.25 + 0.5(1.25 - (0.5)^2 + 1) = 2.25$$
$$y(1.5) \approx w_3 = w_2 + h(w_2 - (t_2)^2 + 1) = 2.25 + 0.5(2.25 - (1.0)^2 + 1) = 3.375$$
$$y(2.0) \approx w_4 = w_3 + h(w_3 - (t_3)^2 + 1) = 3.375 + 0.5(3.375 - (1.5)^2 + 1) = 4.4375$$

Derivation of Euler's Method

Use Taylor's Theorem for $y(t)$:

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$. Since $h = t_{i+1} - t_i$ and $y'(t_i) = f\big(t_i, y(t_i)\big)$,

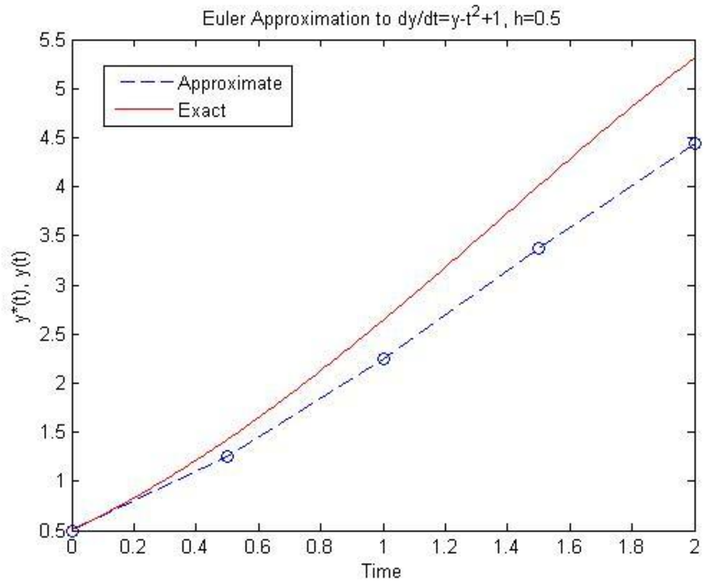
$$y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} y''(\xi_i)$$

Neglecting the remainder term gives Euler's method for $w_i \approx y(t_i)$:

$$w_{i+1} = w_i + hf(t_i, w_i), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Geometric interpretation of Euler's Method

$f(t_i, w_i) \approx y'(t_i) = f(t_i, y(t_i))$ implies $f(t_i, w_i)$ is an approximation to slope of $y(t)$ at t_i.



Error bound

Theorem 5.9 Suppose $D = \{(x, y)\ a \le t \le b\ and - \infty < y < \infty\}$ and that $f(t, y)$ is continuous on D and satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L and that a constant M exists with

$$|y''(t)| \le M, \quad \text{for all } t \in [a, b].$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t, y), \quad a \le t \le b, \ y(a) = \beta,$$

and $w_0, w_1, \cdots, w_n$ as in Euler's method. Then

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right].$$

Example. The solution to the IVP $y' = y - t^2 + 1, \quad 0 \le t \le 2, \ y(0) = 0.5$ was approximated by Euler's method with $h = 0.2$. Find the bound for approximation.

Solution: The exact solution is $y(t) = (t+1)^2 - 0.5e^t$.

$y''(t) = 2 - 0.5e^t$.

So $|y''(t)| \leq 0.5e^2 - 2 = M$ for all $t \in [0, 2]$.

$\left|\frac{\partial}{\partial y}(y - t^2 + 1)\right| = |1| = 1 = L$.

$|y(t_i) - w_i| \leq \frac{(0.2)(0.5e^2-2)}{2(1)}\left[e^{(1)(t_i-0)} - 1\right]$

Hence $|y(0.2) - w_1| \leq 0.1(0.5e^2 - 2)[e^{0.2} - 1]$

$|y(0.4) - w_2| \leq 0.1(0.5e^2 - 2)[e^{0.4} - 1]$ and so on.