5.10 Stability

Consistency and Convergence

Definition. A one-step difference equation with local truncation error $\tau_i(h)$ is said to be ***consistent*** if

$$\lim_{h\to 0}\max_{1\le i\le N}|\tau_i(h)| = 0$$

Definition. A one-step difference equation is said to be ***convergent*** if

$$\lim_{h\to 0}\max_{1\le i\le N}|w_i - y(t_i)| = 0$$

where $y(t_i)$ is the exact solution and w_i is the approximate solution.

Example. To solve $y' = f(t,y), \quad a \le t \le b, \ y(a) = \alpha$. Let $|y''(t)| \le M,$ an $f(t,y)$ be continuous and satisfy a Lipschitz condition with Lipschitz constant L. Show that Euler's method is consistent and convergent.

Solution:

$$|\tau_{i+1}(h)| = |\tfrac{h}{2}y''(\xi_i)| \le \tfrac{h}{2}M$$

$$\lim_{h\to 0}\max_{1\le i\le N}|\tau_i(h)| \le \lim_{h\to 0}\frac{h}{2}M = 0$$

Thus Euler's method is consistent.

By Theorem 5.9,

$$\max_{1\le i\le N}|w_i - y(t_i)| \le \frac{Mh}{2L}[e^{L(b-a)} - 1]$$

$$\lim_{h\to 0}\max_{1\le i\le N}|w_i - y(t_i)| \le \lim_{h\to 0}\frac{Mh}{2L}\left[e^{L(b-a)} - 1\right] = 0$$

Thus Euler's method is convergent.

The rate of convergence of Euler's method is $O(h)$.

Stability

Motivation: How does round-off error affect approximation? To solve IVP $y' = f(t,y), \quad a \le t \le b, \; y(a) = \alpha$ by Euler's method. Suppose δ_i is the round-off error associated with each step.

$$u_0 = \alpha + \delta_0$$

$$u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1} \qquad \text{for each } i = 0, 1, \dots, N-1.$$

Then $|u_i - y(t_i)| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i-a)} - 1\right] + |\delta_0| e^{L(t_i-a)}$. Here $|\delta_i| < \delta$.

$$\lim_{h\to 0}\left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty.$$

Stability: small changes in the initial conditions produce correspondingly small changes in the subsequent approximations. The one-step method is **stable** if there is a constant K and a step size $h_0 > 0$ such that the difference between two solutions w_i and $\tilde{w}_i$ with initial values α and $\tilde{\alpha}$ respectively, satisfies $|w_i - \tilde{w}_i| < K|\alpha\text{-}\tilde{\alpha}|$ whenever $h < h_0$ and $nh \le b - a$.

Convergence of One-Step Methods

Theorem. Suppose the IVP $y' = f(t,y), \quad a \le t \le b, \; y(a) = \alpha$ is approximated by a one-step difference method in the form

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h\phi(t_i, w_i, h) \qquad \text{where } i = 0, 2, \dots N.$$

Suppose also that $h_0 > 0$ exists and $\phi(t, w, h)$ is continuous with a Lipschitz condition in w with constant L on D, then

$$D = \{(t, w, h) |\; a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}.$$

(1) The method is ***stable***;

(2) The method is ***convergent*** if and only if it is ***consistent***:

$$\phi(t, w, 0) = f(t, y), \qquad \text{for all } a \le t \le b$$

(3) If τ exists s.t. $|\tau_i(h)| \le \tau(h)$ when $0 \le h \le h_0$, then

$$|w_i - y(t_i)| \le \frac{\tau(h)}{L} e^{L(t_i-a)}.$$

Example. Show modified Euler method $w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right)$ is stable and convergent.

Solution $\phi(t, w, h) = \frac{1}{2}f(t, w) + \frac{1}{2}f(t, w + hf(t, w))$. Suppose $f(t, w)$ satisfied a Lipschitz condition on $\{(t, w) \mid a \le t \le b, \text{and} -\infty < w < \infty\}$ with Lipschitz constant L.

We next show that $\phi(t, w, h)$ satisfies a Lipschitz condition in w.

$$|\phi(t, w_1, h) - \phi(t, w_2, h)| = \frac{1}{2}\left|f(t, w_1) + f\left(t, w_1 + hf(t, w_1)\right) - f(t, w_2) - f\left(t, w_2 + hf(t, w_2)\right)\right| =$$

$$\frac{1}{2}\left|f(t, w_1) - f(t, w_2) + f\left(t, w_1 + hf(t, w_1)\right) - f\left(t, w_2 + hf(t, w_2)\right)\right|$$

$$\le \frac{1}{2}|f(t, w_1) - f(t, w_2)| + \frac{1}{2}\left|f\left(t, w_1 + hf(t, w_1)\right) - f\left(t, w_2 + hf(t, w_2)\right)\right|$$

$$\le \frac{1}{2}L|w_1 - w_2| + \frac{1}{2}L\left|w_1 + hf(t, w_1) - \left(w_2 + hf(t, w_2)\right)\right| \le L|w_1 - w_2| + \frac{1}{2}Lh|f(t, w_1) - f(t, w_2)|$$

$$\le L|w_1 - w_2| + \frac{1}{2}hL^2|w_1 - w_2| = |w_1 - w_2|(L + \frac{1}{2}hL^2)$$

Therefore, $\phi(t, w, h)$ satisfies a Lipschitz condition in w with constant $(L + \frac{1}{2}hL^2)$ on $\{(t, w, h) \mid a \le t \le b, -\infty < w < \infty, and\ h < \ h_0\}$.

Also, if $f(t, w)$ is continuous on $\{(t, w) \mid a \le t \le b, \text{and} -\infty < w < \infty\}$, then $\phi(t, w, h)$ is continuous on $\{(t, w, h) \mid a \le t \le b, -\infty < w < \infty, and\ h < \ h_0\}$.

So, the modified Euler method is stable.

Moreover,

$$\phi(t, w, 0) = \frac{1}{2}f(t, w) + \frac{1}{2}f(t, w + 0f(t, w)) = f(t, w).$$

This shows that the method is consistent, and the method is convergent.

The local truncation error of modified Euler method is $O(h^2)$. So $|y(t_i) - w_i| = O(h^2)$ by part (iii) of the theorem.

Multi-Step Methods

Definition. A m-step multistep is **consistent** if $\lim_{h\to 0}|\tau_i(h)| = 0$, for all $i = m, m+1, \ldots, N$ and $\lim_{h\to 0}|\alpha_i - y(t_i)| = 0$, for all $i = 1,2,\ldots, m-1$.

Theorem. Suppose the IVP $y' = f(t,y), a \le t \le b,\ y(a) = \alpha$ is approximated by an explicit Adams predictor-corrector method with an m-step Adams-Bashforth predictor equation $w_{i+1} = w_i + h[b_{m-1}f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})]$ with local truncation error $\tau_{i+1}(h)$ and an (*m*-*1*)-step implicit Adams-Moulton corrector equation $w_{i+1} = w_i + h[\tilde{b}_{m-1}f(t_i, w_i) + \cdots + \tilde{b}_0 f(t_{i+2-m}, w_{i+2-m})]$ with local truncation error $\tilde{\tau}_{i+1}(h)$. In addition, suppose that $f(t,y)$ and $f_y(t,y)$ are continuous on $\{(t,y)\,|a \le t \le b, \text{and} -\infty < y < \infty\}$ and that $f_y(t,y)$ is bounded. Then the local truncation error $\sigma_{i+1}(h)$ of the predictor-corrector method is

$$\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h)\tilde{b}_{m-1}f_y(t_{i+1}, \theta_{i+1})$$

where θ_{i+1} is a number between zero and $h\tau_{i+1}(h)$.

Moreover, there exist constant k_1 and k_2 such that

$$|w_i - y(t_i)| \le \left[\max_{0\le j\le m-1}\left|w_j - y(t_j)\right| + k_1\sigma(h)\right]e^{k_2(t_i - a)}$$

where $\sigma(h) = \max_{m\le j\le N}\left|\sigma_j(h)\right|$.

Example. Consider the IVP $y' = 0, \quad 0 \le t \le 10,\ y(0) = 1$, which is solved by $w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-1}))$. If in each step, there is a round-off error ε, and $w_1 = 1 + \varepsilon$. Find out how error propagates with respect to time.

Solution: $w_2 = -4(1+\varepsilon) + 5(1) = 1 - 4\varepsilon$

$$w_3 = -4(1-\varepsilon) + 5(1+\varepsilon) = 1 + 21\varepsilon$$

$$w_4 = -4(1+21\varepsilon) + 5(1-4\varepsilon) = 1 - 104\varepsilon.$$

Definition. Consider to solve the IVP: $y' = f(t,y), \quad a \le t \le b,\ y(a) = \alpha$. by an m-step multistep method

$$\begin{aligned} w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ &h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1}f(t_i, w_i) + \cdots \\ &+ b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned}$$

The **characteristic polynomial** of the method is given by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

Remark:

(1) The **characteristic polynomial** can be viewed as derived by solving $y' = 0,\ y(a) = \alpha$ using the m-step multistep method.

(2) If λ is a root of the characteristic polynomial, then $w_i = \lambda^i$ for each i is a solution to $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$.

This is because $\lambda^{i+1} - a_{m-1}\lambda^i - a_{m-2}\lambda^{i-1} - \cdots - a_0\lambda^{i+1-m} = \lambda^{i+1-m}(\lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0) = 0$

(3) If $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$ are distinct zeros of the **characteristic polynomial,** solution to $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$ can be represented by $w_i = \sum_{j=1}^{m} c_j \lambda_j^i$ for some unique constants $c_1, \dots, c_m$.

(4) $w_i = \alpha$ is a solution to $w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$, this is because $y(t) = \alpha$ is the exact solution to $y' = 0,\ y(a) = \alpha$.

(5) From (4), $0 = \alpha - a_{m-1}\alpha - a_{m-2}\alpha - \cdots - a_0\alpha = \alpha[1 - a_{m-1} - a_{m-2} - \cdots - a_0]$. Compare this with definition of characteristic polynomial, this shows that $\lambda = 1$ is one of the zeros of the characteristic polynomial.

(6) Let $\lambda_1 = 1$ and $c_1 = \alpha$, solution to $y' = 0,\ y(0) = \alpha$ is expressed as $w_i = \alpha + \sum_{j=2}^{m} c_j \lambda_j^i$. This means that $c_2, \dots, c_m$ would be zero if all the calculations were exact. However, $c_2, \dots, c_m$ are not zero in practice due to round-off error.

(*) The stability of a multistep method with respect to round-off error is dictated by magnitudes of zeros of the characteristic polynomial. If $|\lambda_j| > 1$ for any of $\lambda_2, \lambda_3, \dots, \lambda_m$, the round-off error grows exponentially.

Example. Consider stability of $w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-1}))$ for solving $y' = 0,\quad 0 \le t \le 10,\ y(0) = 1$.

Solution: The difference eq. is $w_{i+1} = -4w_i + 5w_{i-1}$ with initial condition $w_0 = 1, w_1 = 1 + \delta$. δ is due to round-off error.

The characteristic polynomial is $P(\lambda) = \lambda^2 + 4\lambda - 5$.

The general solution to the difference eq. is $w_i = c_1(1)^i + c_2(-5)^i$.

Using the initial condition: $c_1 + c_2 = 1,\quad c_1 - 5c_2 = 1 + \delta$. This implies $c_1 = 1 + \frac{\delta}{6},\quad c_2 = -\frac{\delta}{6}$.

The solution to difference eq. then is: $w_i = (1 + \frac{\delta}{6})(1)^i + (-\frac{\delta}{6})(-5)^i$.

Remark: the term $(-\frac{\delta}{6})(-5)^i$ shows how round-off error grows.

Definition. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the roots of the **characteristic equation**

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0 = 0$$

associated with the *m*-step multistep method

$$\begin{aligned} w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots \\ + b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned}$$

If $|\lambda_i| \leq 1$ and all roots with absolute value 1 are simple roots, then the difference equation is said to satisfy the **root condition**.

Stability of multistep method

Definition of stability of multistep method.

1) Methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation with magnitude one are called **strongly stable**.
2) Methods that satisfy the root condition and have more than one distinct roots with magnitude one are called **weakly stable**.
3) Methods that do not satisfy the root condition are called **unstable**.

Example. Show 4th order Adams-Bashforth method

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})]$$

is strongly stable.

Solution: The characteristic equation of the 4th order Adams-Bashforth method is

$$P(\lambda) = \lambda^4 - \lambda^3 = 0$$

$$0 = \lambda^4 - \lambda^3 = \lambda^3(\lambda - 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$.

Therefore $P(\lambda)$ satisfies root condition and the method is strongly stable.

Example. Show 4th order Miline's method

$$w_{i+1} = w_{i-3} + \frac{4h}{3}[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})]$$

is weakly stable.

Solution: The characteristic equation $P(\lambda) = \lambda^4 - 1 = 0$

$$0 = \lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1)$$

$P(\lambda)$ has roots $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i.$

All roots have magnitude one. So the method is weakly stable.

Theorem. A multistep method

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots$$
$$+ b_0 f(t_{i+1-m}, w_{i+1-m})],$$

is stable **if and only if** it satisfies the root condition. If it is also consistent, then it is stable **if and only if** it is convergent.