## 5.11 Stiff Differential Equation

**Example.** The initial-value problem $y' = -30y, \quad 0 \le t \le 1.5, \; y(0) = \frac{1}{3}$ has exact solution $y(t) = \frac{1}{3}e^{-30t}$. Use Euler's method and 4-stage Runge-Kutta method to solve with step size $h = 0.1$ respectively.

**Solution: Euler's method**

$$w_{i+1} = (1 - 30h)w_i = (1 - 30h)^2 w_{i-1} = \cdots = (1 - 30h)^{i+1} w_0$$

If $h > \frac{1}{15}$, then $|1 - 30h| > 1$, and $(1 - 30h)^{i+1}$ grows geometrically, in contrast to the true solution.

**Remark**: The local truncation error for Euler's method is small even for $h > \frac{1}{15}$, but the algorithm fails because of instability (but a different kind of instability than discussed in **Sec. 5.10,** because this one occurs only for $h > \frac{1}{15}$).

**Facts:**

**1)** A stiff differential equation is numerically unstable unless the step size is extremely small.

**2)** Stiff differential equations are characterized as those whose exact solution has a term of the form $e^{-ct}$, where $c$ is a large positive constant.

**3)** Large derivatives of $e^{-ct}$ give error terms that are dominating the solution.

**Definition.** The *test equation* is said to be $y' = \lambda y, \qquad y(0) = \alpha,$ where $\lambda < 0$

The test equation has exact solution $y(t) = \alpha e^{\lambda t}$.

**Euler's Method for Test Equation**

$$w_0 = \alpha$$

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j = (1 + h\lambda)(1 + h\lambda)w_{j-1} = \cdots = (1 + h\lambda)^{j+1}\alpha \qquad \text{for } j = 0,1,\ldots,N-1$$

The absolute error is $|y(t_j) - w_j| = |e^{jh\lambda} - (1 + h\lambda)^j||\alpha| = |(e^{h\lambda})^j - (1 + h\lambda)^j||\alpha|$

So 1) the accuracy is determined by how well $(1 + h\lambda)$ approximate $e^{h\lambda}$.

2) $(e^{h\lambda})^j$ decays to zero as $j$ increases. $(1 + h\lambda)^j$ will decay to zero only if $|1 + h\lambda| < 1$. This implies that

$-2 < h\lambda < 0$ or $h < 2/|\lambda|$.

**Note**: Euler's method is expected to be stable for the test equation only if the step size $h < 2/|\lambda|$.
Also, **define** $Q(h\lambda) = 1 + h\lambda$ for Euler's method, then $w_{j+1} = Q(h\lambda)w_j$

Now suppose a round-off error $\delta_0$ is introduced in the initial condition for Euler's method

$$w_0 = \alpha + \delta_0$$

$$w_j = (1 + h\lambda)^j(\alpha + \delta_0)$$

At the jth step, the round-off error is $\delta_j = (1 + h\lambda)^j\delta_0$.

So with $\lambda < 0$, the condition for control of the growth of round-off error is the same as the condition for controlling the absolute error $|1 + h\lambda| < 1$.

**Nth-order Taylor Method for Test Equation**

Applying the nth-order Taylor method to the test equation leads to

$$\left|1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n\right| < 1$$

to have stability. Also, define $Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \cdots + \frac{1}{n!}(h\lambda)^n$ for a nth-order Taylor method, i.e., $w_{j+1} = Q(h\lambda)w_j$.

**Multistep Method for Test Equation**

Apply a multistep method to the test equation:

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m}$$
$$h\lambda[b_m w_{i+1} + b_{m-1}w_i + \cdots$$
$$+b_0 w_{i+1-m}],$$

or

$$(1 - h\lambda b_m)w_{i+1} - (a_{m-1} - h\lambda b_{m-1})w_i - \cdots - (a_0 - h\lambda b_0)w_{i+1-m} = 0$$

Define the associated **characteristic polynomial** to this difference equation

$$Q(z, h\lambda) = (1 - h\lambda b_m)z^m - (a_{m-1} - h\lambda b_{m-1})z^{m-1} - \cdots - (a_0 - h\lambda b_0).$$

Let $\beta_1, \beta_2, \dots, \beta_m$ be the zeros of the **characteristic polynomial** to the difference equation.

Then $c_1, c_2, \dots, c_m$ exist with

$$w_i = \sum_{k=1}^{m} c_k (\beta_k)^i, \qquad \text{for} \quad i = 0, \dots N$$

and $|\beta_k| < 1$ is required for stability.

**Region of Stability**

**Definition.** The **region *R* of absolute stability** for a one-step method is $R = \{h\lambda \in C \mid |Q(h\lambda)| < 1\}$, and for a multistep method, it is $R = \{h\lambda \in C \mid |\beta_k| < 1, \text{ for all zeros } \beta_k \text{ of } Q(z, h\lambda)\}$.

A numerical method is said to be A-stable if its region $R$ of absolute stability contains the entire left half-plane.

The only A-stable multistep method is **implicit Trapezoidal method**.

$$w_0 = \alpha$$

$$w_{j+1} = w_j + \frac{h}{2}\left[f(t_j, w_j) + f(t_{j+1}, w_{j+1})\right], \qquad \text{for} \quad 0 \le j \le N - 1.$$

The A-stable **implicit backward Euler method**.

$$w_0 = \alpha$$
$$w_{j+1} = w_j + hf(t_{j+1}, w_{j+1}), \qquad \text{for} \quad 0 \le j \le N-1.$$

Remark: The technique commonly used for stiff systems is implicit methods.

**Example**. Show Backward Euler method has $Q(h\lambda) = \frac{1}{1-h\lambda}$.

**Solution**: $w_{j+1} = w_j + h\lambda w_{j+1}$

$$w_{j+1} = \frac{1}{1-h\lambda} w_j = (\frac{1}{1-h\lambda})^2 w_{j-1} = \cdots = (\frac{1}{1-h\lambda})^{j+1} w_0$$

Stability implies $|\frac{1}{1-h\lambda}| < 1$.