5.4 Runge-Kutta Methods

Motivation: Obtain high-order accuracy of Taylor's method without knowledge of derivatives of $f(t,y)$.

Theorem 5.13(Taylor's Theorem in Two Variables) Suppose $f(t,y)$ and partial derivative up to order $n+1$ continuous on
$D=\{(t,y)|a\ \le t\le b, c\ \le y\le d\}$, let $(t_0,y_0)\in D$. For $(t,y)\in D$, there is $\xi\in[t,t_0]$ and $\mu\in[y,y_0]$ with

$$f(t,y)=P_n(t,y)+R_n(t,y).$$

Here

$$\begin{aligned}P_n(t,y)=f(t_0,y_0)&+\left[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0)+(y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)\right]\\&+\left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0)+(t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t\partial y}(t_0,y_0)+\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right]\\&+\left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]\end{aligned}$$

$$R_n(t,y)=\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(t_0,y_0)$$

$P_n(t,y)$ is the nth Taylor polynomial in two variables.

<u>Derivation of Runge-Kutta method of order two</u>

1. Determine a_1, α_1, β_1 such that

$$a_1 f(t+\alpha_1,y+\beta_1)\approx f(t,y)+\frac{h}{2}f'(t,y)=T^{(2)}(t,y)\text{ with }O(h^2)\text{ error.}$$

Notice $f'(t,y)=\frac{df(t,y(t))}{dt}=\frac{\partial f}{\partial t}\big(t,y(t)\big)+\frac{\partial f}{\partial y}\big(t,y(t)\big)\cdot y'(t)=\frac{\partial f}{\partial t}\big(t,y(t)\big)+\frac{\partial f}{\partial y}\big(t,y(t)\big)\cdot f\big(t,y(t)\big)$

We have $T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}\big(t,y(t)\big) + \frac{h}{2}\frac{\partial f}{\partial y}\big(t,y(t)\big)\cdot f\big(t,y(t)\big)$ (1)

2. Expand $a_1 f(t+\alpha_1, y+\beta_1)$ in 1st degree Taylor polynomial:

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1(t+\alpha_1, y+\beta_1) \qquad (2)$$

3. Match coefficients of equation (1) and (2) gives

$$a_1 = 1, \qquad a_1\alpha_1 = \frac{h}{2}, \qquad a_1\beta_1 = \frac{h}{2} f\big(t,y(t)\big)$$

with unique solution

$$a_1 = 1, \qquad \alpha_1 = \frac{h}{2}, \qquad \beta_1 = \frac{h}{2} f\big(t,y(t)\big)$$

4. This gives

$$T^{(2)}(t,y) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t,y(t)\big)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t,y(t)\big)\right)$$

with $R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f\big(t,y(t)\big)\right) = O(h^2)$

Midpoint Method (one of Runge-Kutta methods of order two)

Consider the IVP

$$y' = f(t,y), \qquad a \le t \le b, \quad y(a) = \beta.$$

with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right), \qquad \text{for each } i = 0,1,2,\cdots,N-1.$$

Local truncation error is $O(h^2)$.

Two stage formula:

$$w_0 = \beta$$
$$k_1 = f(t_i, w_i)$$
$$k_2 = f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}k_1\right)$$
$$w_{i+1} = w_i + hk_2$$

Example 2. Use the Midpoint method with $N = 10, h = 0.2,\ t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$

Modified Euler Method (Another Runge-Kutta methods of order two)

Consider the IVP

$$y' = f(t, y), \quad a \le t \le b, \ y(a) = \beta.$$

with step size $h = \frac{b-a}{N}$.

$$w_0 = \beta$$
$$w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right), \qquad \text{for each } i = 0, 1, 2, \cdots, N-1.$$

Local truncation error is $O(h^2)$.

Two stage formula:

$$w_0 = \beta$$
$$k_1 = f(t_i, w_i)$$
$$k_2 = f(t_{i+1}, w_i + hk_1)$$
$$w_{i+1} = w_i + \frac{h}{2}[k_1 + k_2]$$

Example. Use the Modified Euler method with $N = 10, h = 0.2,\ t_i = 0.2i$ and $w_0 = 0.5$ to solve the IVP

$$y' = y - t^2 + 1, \qquad 0 \le t \le 2, \quad y(0) = 0.5.$$

Heun's Method (Runge-Kutta Method of order three)

Idea: Approximate $T^3(t, y)$ with $O(h^3)$ error by $f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y)))$

$$w_0 = \beta$$

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3f(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right))\right), \qquad \text{for each } i$$

$$= 0, 1, 2, \cdots, N - 1.$$

Runge-Kutta Method of order four

$$w_0 = \beta$$

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2} k_1\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2} k_2\right)$$

$$k_4 = hf(t_{i+1}, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad \text{for each } i = 0, 1, 2, \cdots, N - 1.$$