7.3 The Jacobi and Gauss-Seidel Iterative Methods

The Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{gathered}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$\begin{gathered} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n) \\ \vdots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1}) \end{gathered}$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation*, $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approximation's x-vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots x_n^{(k)}\right)^t, \quad k = 1,2,3, \dots$

The Jacobi Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(\boldsymbol{k})}$ from $\boldsymbol{x}^{(\boldsymbol{k}-\boldsymbol{1})}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}\left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \qquad \text{for } i = 1,2,,\dots n$$

Example. Apply the Jacobi method to solve

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, rewrite the system

$$x_1 = \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$

$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

$$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

$$x_2^{(1)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3^{(1)} = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

Continue iteration, we obtain

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192			
$x_2^{(k)}$	0.000	0.222	0.203	0.328			
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416			

When to stop: 1. $\frac{||\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}||}{||\boldsymbol{x}^{(k)}||} < \varepsilon$; or 2 $\left|\left|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right|\right| < \varepsilon$. Here ε is a given small number .

<u>The Jacobi Method in Matrix Form</u>

Consider to solve an $n \times n$ size system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ -a_{21} & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = D - L - U$$

$A\boldsymbol{x} = \boldsymbol{b}$ is transformed into $(D - L - U)\boldsymbol{x} = \boldsymbol{b}$

$$D\boldsymbol{x} = (L + U)\boldsymbol{x} + \boldsymbol{b}$$

Assume D^{-1} exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$\boldsymbol{x} = D^{-1}(L+U)\boldsymbol{x} + D^{-1}\boldsymbol{b}$$

The matrix form of Jacobi iterative method is

$$\boldsymbol{x}^{(k)} = D^{-1}(L+U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} \qquad k = 1,2,3,\ldots$$

Define $T_j = D^{-1}(L+U)$ and $\boldsymbol{c} = D^{-1}\boldsymbol{b}$, Jacobi iteration method can also be written as

$$\boldsymbol{x}^{(k)} = T_j\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad k = 1,2,3,\ldots$$

<u>Numerical Algorithm of Jacobi Method</u>

Input: $A = [a_{ij}]$, $\boldsymbol{b}, \boldsymbol{XO} = \boldsymbol{x}^{(0)}$, tolerance TOL, maximum number of iterations N.

Step 1 Set $k = 1$

Step 2 while ($k \le N$) do Steps 3-6

Step 3 For for $i = 1,2,,\ldots n$

$$x_i = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}(-a_{ij}\boldsymbol{XO}_j) + b_i\right],$$

Step 4 If $||\boldsymbol{x} - \boldsymbol{XO}|| < TOL$, then OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1,2,,\ldots n$

Set $\boldsymbol{XO}_i = x_i$.

Step 7 OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

Another stopping criterion in Step 4: $\frac{||\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}||}{||\boldsymbol{x}^{(k)}||}$

The Gauss-Seidel Method

Main idea of Gauss-Seidel

With the Jacobi method, only the values of $x_i^{(k)}$ obtained in the kth iteration are used to compute $x_i^{(k+1)}$. With the Gauss-Seidel method, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example, once we have computed $x_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(k+1)}$, and so on.

Example. Derive iteration equations for the Jacobi method and Gauss-Seidel method to solve

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$x_1^{(k)}$	0.000	-0.200	0.167				
$x_2^{(k)}$	0.000	0.156	0.334				
$x_2^{(k)}$	0.000	-0.508	-0.429				

The Gauss-Seidel Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(\boldsymbol{k})}$ from $\boldsymbol{x}^{(\boldsymbol{k-1})}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right], \qquad \text{for } i = 1,2,,\ldots n$$

Namely,

$$a_{11}x_1^{(k)} = -a_{12}x_2^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1$$
$$a_{22}x_2^{(k)} = -a_{21}x_1^{(k)} - a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$$
$$\vdots$$
$$a_{nn}x_n^{(k)} = -a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots + b_n$$

Matrix form of Gauss-Seidel method.

$$(D-L)\boldsymbol{x}^{(\boldsymbol{k})} = U\boldsymbol{x}^{(k-1)} + \boldsymbol{b}$$

$$\boldsymbol{x}^{(\boldsymbol{k})} = (D-L)^{-1}U\boldsymbol{x}^{(k-1)} + (D-L)^{-1}\boldsymbol{b}$$

Define $T_g = (D-L)^{-1}U$ and $\boldsymbol{c}_g = (D-L)^{-1}\boldsymbol{b}$, Gauss-Seidel method can be written as

$$\boldsymbol{x}^{(k)} = T_g\boldsymbol{x}^{(k-1)} + \boldsymbol{c}_g \qquad k = 1,2,3,\ldots$$

<u>Numerical Algorithm of Gauss-Seidel Method</u>

Input: $A = [a_{ij}]$, $\boldsymbol{b}, \boldsymbol{XO} = \boldsymbol{x}^{(0)}$, tolerance *TOL*, maximum number of iterations N.

Step 1 Set $k = 1$

Step 2 while ($k \le N$) do Steps 3-6

Step 3 For for $i = 1,2,,\ldots n$

$$x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j) - \sum_{j=i+1}^{n}(a_{ij}\boldsymbol{XO}_j) + b_i\right],$$

Step 4 If $||\boldsymbol{x} - \boldsymbol{XO}|| < TOL$, then OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1,2,,\ldots n$

Set $\boldsymbol{XO}_i = x_i$.

Step 7 OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

Convergence theorems of the iteration methods

Let the iteration method be written as

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad \text{for each } k = 1,2,3,\ldots$$

Lemma 7.18 If the spectral radius satisfies $\rho(T) < 1$, then $(I-T)^{-1}$ exists, and

$$(I-T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$$

Theorem 7.19 For any $\boldsymbol{x}^{(0)} \in R^n$, the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad \text{for each } k \geq 1$$

converges to the unique solution of $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{c}$ if and only if $\rho(T) < 1$.

Proof (only show $\rho(T) < 1$ is sufficient condition)

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} = T\left(T\boldsymbol{x}^{(k-2)} + \boldsymbol{c}\right) + \boldsymbol{c} = \cdots = T^k\boldsymbol{x}^{(0)} + (T^{k-1} + \cdots + T + I)\boldsymbol{c}$$

Since $\rho(T) < 1$, $\lim_{k\to\infty} T^k\boldsymbol{x}^{(0)} = \boldsymbol{0}$

$$\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{0} + \lim_{k\to\infty}\left(\sum_{j=0}^{k-1} T^j\right)\boldsymbol{c} = (I-T)^{-1}\boldsymbol{c}$$

Corollary 7.20 If $||T|| < 1$ for any natural matrix norm and $\boldsymbol{c}$ is a given vector, then the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$ converges, for any $\boldsymbol{x}^{(0)} \in R^n$, to a vector $\boldsymbol{x} \in R^n$, with $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{c}$**,** and the following error bound hold:

(i) $||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| \leq ||T||^k ||\boldsymbol{x}^{(0)} - \boldsymbol{x}||$

(ii) $||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| \leq \frac{||T||^k}{1-||T||} ||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}||$

Theorem 7.21 If A is strictly diagonally dominant, then for any choice of $\boldsymbol{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ that converges to the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Rate of Convergence

Corollary 7.20 (i) implies $||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| \approx \rho(T)^k ||\boldsymbol{x}^{(0)} - \boldsymbol{x}||$

Theorem 7.22 (Stein-Rosenberg) If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} \geq 0$, for each $i = 1,2,\ldots,n$, then one and only one of following statements holds:

(i) $0 \leq \rho(T_g) < \rho(T_j) < 1$;
(ii) $1 < \rho(T_j) < \rho(T_g)$;
(iii) $\rho(T_j) = \rho(T_g) = 0$;
(iv) $\rho(T_j) = \rho(T_g) = 1$.

Simple iteration

Assume there is an initial guess $\boldsymbol{x}^{(0)}$ for the solution to $A\boldsymbol{x} = \boldsymbol{b}$. If one could compute the *error* $\boldsymbol{e}^{(0)} = A^{-1}\boldsymbol{b} - \boldsymbol{x}^{(0)}$, then one could find the solution $\boldsymbol{x} = \boldsymbol{x}^{(0)} + \boldsymbol{e}^{(0)}$. However, $A^{-1}\boldsymbol{b}$ is expensive to compute.

Instead, let's compute the *residual* $\boldsymbol{r}^{(0)} = \boldsymbol{b} - A\boldsymbol{x}^{(0)}$. Note that $\boldsymbol{r}^{(0)} = A\boldsymbol{e}^{(0)}$.

If M is a matrix with the property that $M^{-1}A$ approximates the identity, yet $\boldsymbol{r}^{(0)} = M\boldsymbol{z}^{(0)}$ is easy to solve for $\boldsymbol{z}^{(0)}$, one could approximate the *error* by $M^{-1}\boldsymbol{r}^{(0)}$.

Thus, one solve $M\boldsymbol{z}^{(0)} = \boldsymbol{r}^{(0)}$ for $\boldsymbol{z}^{(0)}$, which approximates $\boldsymbol{e}^{(0)}$, and then replace $\boldsymbol{x}^{(0)}$ by $\boldsymbol{x}^{(1)} \equiv \boldsymbol{x}^{(0)} + \boldsymbol{z}^{(0)}$. Repeat this process with $\boldsymbol{x}^{(1)}$.

Simple iteration algorithm

Given an initial guess $\boldsymbol{x}^{(0)}$, compute $\boldsymbol{r}^{(0)} = \boldsymbol{b} - A\boldsymbol{x}^{(0)}$, and solve $\boldsymbol{r}^{(0)} = M\boldsymbol{z}^{(0)}$ for $\boldsymbol{z}^{(0)}$.

for $k = 1,2,\ldots$

Set $\boldsymbol{x}^{(k)} \equiv \boldsymbol{x}^{(k-1)} + \boldsymbol{z}^{(k-1)}$.

Compute $\boldsymbol{r}^{(k)} = \boldsymbol{b} - A\boldsymbol{x}^{(k)}$.

Solve $M\boldsymbol{z}^{(k)} = \boldsymbol{r}^{(k)}$ for $\boldsymbol{z}^{(k)}$.

Remark:

1. For $M \equiv D$, the method is Jacobi iteration

$$\boldsymbol{x}^{(k)} = D^{-1}(L+U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} = D^{-1}(D - D + L + U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} = D^{-1}(D-A)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} = \boldsymbol{x}^{(k-1)} + D^{-1}\left(\boldsymbol{b} - A\boldsymbol{x}^{(k-1)}\right) \equiv \boldsymbol{x}^{(k-1)} + \boldsymbol{z}^{(k-1)},$$

2. For $M \equiv D - L$, the method is GS.

$$\begin{aligned}\boldsymbol{x}^{(k)} &= (D-L)^{-1}U\boldsymbol{x}^{(k-1)} + (D-L)^{-1}\boldsymbol{b} = (D-L)^{-1}(U - D + L + D - L)\boldsymbol{x}^{(k-1)} + (D-L)^{-1}\boldsymbol{b} \\ &= (D-L)^{-1}(D - L - A)\boldsymbol{x}^{(k-1)} + (D-L)^{-1}\boldsymbol{b} = \boldsymbol{x}^{(k-1)} - (D-L)^{-1}\left(\boldsymbol{b} - A\boldsymbol{x}^{(k-1)}\right) \equiv \boldsymbol{x}^{(k-1)} + \boldsymbol{z}^{(k-1)}\end{aligned}$$