# 8.2 - Orthogonal Polynomials and Least Squares Approximation

# Least Squares Approximation of Functions

## Motivation

Suppose $f \in C[a,b]$, find a polynomial $P_n(x)$ of degree at most $n$ to approximate $f$ such that $\int_a^b (f(x) - P_n(x))^2 \, dx$ is a minimum.

Let polynomial $P_n(x)$ be $P_n(x) = \sum_{k=0}^{n} a_k x^k$ which minimizes the error

$$E \equiv E(a_0, a_1, \ldots, a_n) = \int_a^b \left( f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 dx.$$

The problem is to find $a_0, \cdots, a_n$ that will minimize $E$. The necessary condition for $a_0, \cdots, a_n$ to minimize $E$ is $\frac{\partial E}{\partial a_j} = 0$, which gives the **normal equations**

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} \, dx = \int_a^b x^j f(x) \, dx \quad \text{for } j = 0, 1, \cdots, n$$

**Example** Find the least squares approximating polynomial of degree 2 for $f(x) = \sin \pi x$ on $[0, 1]$.
**Solution** Let $P_2(x) = a_0 + a_1 x + a_2 x^2$.

$$\begin{cases} a_0 \int_0^1 1 dx + a_1 \int_0^1 x dx + a_2 \int_0^1 x^2 dx = \int_0^1 \sin \pi x dx \\ a_0 \int_0^1 x dx + a_1 \int_0^1 x^2 dx + a_2 \int_0^1 x^3 dx = \int_0^1 x \sin \pi x dx \\ a_0 \int_0^1 x^2 dx + a_1 \int_0^1 x^3 dx + a_2 \int_0^1 x^4 dx = \int_0^1 x^2 \sin \pi x dx \end{cases} \tag{1}$$

$$\begin{cases} a_0 + \frac{1}{2} a_1 + \frac{1}{3} a_2 = 2/\pi \\ \frac{1}{2} a_0 + \frac{1}{3} a_1 + \frac{1}{4} a_2 = 1/\pi \\ \frac{1}{3} a_0 + \frac{1}{4} a_1 + \frac{1}{5} a_2 = \frac{\pi^2 - 4}{\pi^3} \end{cases}$$

$a_0 = -0.050465$, $a_1 = 4.12251$, $a_2 = -4.12251$.
Remark: System of linear equations with coefficient matrix being Hilbert matrix $H$ whose entry $H_{jk}$ is defined by $H_{jk} = \int_a^b x^{j+k} dx$ is difficult to solve due to round-off errors.

# Linearly Independent Functions

## Definition

The set of functions $\{\phi_0, \cdots, \phi_n\}$ is said to **linearly independent** on $[a, b]$ if, whenever $c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0, \quad \text{for all } x \in [a, b]$ we have $c_0 = c_1 = \cdots = c_n = 0$. Otherwise the set of functions is said to be **linearly dependent**.

## Theorem

*Suppose that, for each $j = 0, 1, \cdots, n$, $\phi_j(x)$ is a polynomial of degree $j$. Then $\{\phi_0, \cdots, \phi_n\}$ is linearly independent on any interval $[a, b]$.*

## Theorem

*Suppose that $\{\phi_0(x), \cdots, \phi_n(x)\}$ is a collection of linearly independent polynomials in $\prod_n$. Then any polynomial in $\prod_n$ can be written uniquely as a linear combination of $\phi_0(x), \cdots, \phi_n(x)$.*

# Orthogonal Functions

**Definition**

$\{\phi_0, \cdots, \phi_n\}$ is said to be an *orthogonal set of functions* for the interval $[a, b]$ with respect to the weight function $w$ if

$$\int_a^b w(x)\phi_j(x)\phi_k(x)\,dx = \begin{cases} 0, & \text{when } j \neq k, \\ \alpha_k > 0, & \text{when } j = k. \end{cases}$$

If also $\alpha_k = 1$ for each $k = 0, \ldots, n$, the set is *orthonormal*.

The set of *Legendre polynomials* $\{P_n(x)\}$ is orthogonal on $[-1, 1]$ w.r.t. the weight function $w(x) = 1$.

$$P_0(x) = 1, \qquad \alpha_0 = 2$$

$$P_1(x) = x, \qquad \alpha_1 = 2/3$$

$$P_2(x) = x^2 - \frac{1}{3}, \qquad \alpha_2 = 8/45$$

$$P_3(x) = x^3 - \frac{3}{5}x, \qquad \alpha_3 = 8/175$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}, \qquad \alpha_4 = 128/11025$$

# Orthogonal Trigonometric Polynomials

The set of $\{\phi_0, \cdots, \phi_{2n-1}\}$ is orthogonal on $[-\pi, \pi]$ w.r.t. the weight function $w(x) = 1$.

$$\begin{aligned} \phi_0(x) &= 1/2 \ , \\ \phi_k(x) &= cos(kx) \ , \ \text{for each } k = 1, 2, \cdots, n, \\ \phi_{n+k}(x) &= sin(kx) \ , \ \text{for each } k = 1, 2, \cdots, n-1 \ . \end{aligned}$$

# Approximation by Orthogonal Functions

Let $\{\phi_0, \cdots, \phi_n\}$ be an *orthogonal set of functions* for the interval $[a, b]$ with respect to the weight function $w$.
Given $f \in C[a, b]$, we seek an approximation

$$P(x) = \sum_{i=0}^{n} a_k \phi_k$$

to minimize the error

$$E \equiv E(a_0, a_1, \ldots, a_n) = \int_a^b w \left( f(x) - \sum_{k=0}^{n} a_k \phi_k \right)^2 dx \ .$$

Computing $\frac{\partial E}{\partial a_j} = 0$ gives the **normal equations**

$$0 = 2 \int_a^b w(x) \left( f(x) - \sum_{k=0}^{n} a_k \phi_k \right) \phi_j dx \quad \text{for } j = 0, 1, \cdots, n \qquad (2)$$

Simplying Eq. (2) gives:

$$\int_a^b w(x)\phi_j(x)f(x)\,dx = a_j \int_a^b w(x)[\phi_j(x)]^2\,dx\ , \quad \text{for } j = 0, 1, \cdots, n\ .$$

## Theorem

*If* $\{\phi_0, \ldots, \phi_n\}$ *is orthogonal on* $[a, b]$*, then the least squares approximation to* $f$ *on* $[a, b]$ *is* $P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$ *where*

$$a_k = \frac{\int_a^b w(x)\phi_k(x)f(x)\,dx}{\int_a^b w(x)[\phi_k(x)]^2\,dx} = \frac{1}{\alpha_k}\int_a^b w(x)\phi_k(x)f(x)\,dx, \ \text{for } k = 0, 1, \cdots, n.$$

*And*

$$\alpha_k = \int_a^b w(x)[\phi_k(x)]^2\,dx\ .$$